Seminarie analyse en topologie

Localic topoi

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1 Introduction

In algebraic geometry one studies spaces as follows: Let **Spaces** be a category of spaces and **Alg** be some algebraic category. Assume $\mathcal{R}$ is an algebraic object in **Spaces**, then has $\mathbf{Spaces}(S, \mathcal{R})$ a natural algebraic structure such that $\mathbf{Spaces}(S, \mathcal{R}) \in \mathbf{Alg}$. So we get a (contravariant) functor

$$\mathbf{Spaces}(-, \mathcal{R}) : \mathbf{Spaces} \to \mathbf{Alg} : S \mapsto \mathbf{Spaces}(S, \mathcal{R}).$$

So for example one considers $\mathbf{Spaces} := \mathbf{Schemes}$, the category of schemes and $\mathbf{Alg} := \mathbf{CRing}$, the category of commutative unital rings.
The choice of **Alg** and $\mathcal{R}$ is dependent on which properties of the spaces are needed to be studied. In this paper we are concerned with studying the open subsets of a space. In order to achieve this, one can choose $\mathcal{R}$ as the *Sierpinski space* $\mathcal{S} := (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\})$. The reason for this is that $\mathcal{S}$ classifies the open subsets in the following way: Let $S \in \mathbf{Top}$ a top. space, then for each $U \subseteq S$ open, there exists a unique continuous map $\chi_U : S \to \mathcal{S}$ such that the following diagram commutes

$$\begin{array}{ccc} U & \hookrightarrow & S \\ \downarrow & & \downarrow \chi_U \\ \{\star\} & \xrightarrow{1} & \mathcal{S} \end{array}$$

We call χ_U is the **characteristic map** of U in X and is given by $\chi_U(x) = 1 \iff x \in U$. Moreover, this map is universal in the sense that if $\chi_U(V) = 1$, we have $V \subseteq U$. So U is the *biggest* open subset of S for which the diagram commutes. Consequently, if we denote by $\mathcal{O}(S)$ the topology on S, i.e. the set of open subsets, we have a bijection between $\mathbf{Top}(S, \mathcal{S})$ and $\mathcal{O}(S)$. So in the spirit of algebraic geometry, the question remains whether we can put some algebraic structure on $\mathcal{O}(S)$. As algebraic structure, we use its poset structure where the algebraic operations are the suprema and infima.
Another reason that an algebraic geometer is interested in $\mathcal{O}(S)$, is because the sheaf property is/can be expressed using purely (the poset structure of) the open sets of S. So in order to test whether a property holds for arbitrary (Grothendieck) topoi, one can first test it for topoi on topological spaces and (afterwards) on *nice* poset categories which generalize $\mathcal{O}(S)$.
So in the rest of the paper we first study those posets which behave like $\mathcal{O}(S)$, which will be called *locales*. Then we study those grothendieck topoi, where the underlying category is a locale.

2 Frames and Heyting algebras

As mentioned in the introduction, we would like to describe the poset structure of the open sets of a topological space S. The axioms of a topology says precisely that, as a poset, $\mathcal{O}(S)$ has arbitrary suprema, finite infima which contains a bottom and top element. Since the suprema and (finite) infima are the union and intersection we automatically have some distributivity property which leads precisely to the definition of a frame:

Definition 1. *A **frame** is a bounded lattice F with all finite infima, all (arbitrary) suprema which satisfies the following distributivity axiom:*

$$x \wedge \bigvee_i y_i = \bigvee_i (x \wedge y_i), \quad \forall x, y_i \in F.$$

In the definition of a frame, we only have finite infima because an arbitrary intersection of opens is not necessarily open, but one can take the interior so that $\mathcal{O}(S)$ indeed has all infima. This holds also in every frame F by the formula $\bigwedge K = \bigvee\{x \in F | \forall y \in K : x \leq y\}$ (for $K \subseteq F$). We can therefore also ask whether we shouldn't demand the following distributivity rule:

$$x \vee \bigwedge_i y_i = \bigwedge_i (x \vee y_i) .$$

This does in general not hold:

Example 1. In $\mathcal{O}(S)$, the (arbitrary) infima is given by:

$$\bigwedge_i U_i = int\left(\bigcap_i U_i\right).$$

Consider $\mathbb{R}$ with the Euclidean topology and let

$$U := \mathbb{R} \setminus \{0\}, \quad V_\epsilon := (-\epsilon, \epsilon), \epsilon > 0.$$

So $\bigwedge_{\epsilon>0} V_\epsilon = int(\{0\}) = \emptyset$ and $U \vee V_\epsilon = \mathbb{R}$, thus

$$U \vee \bigwedge_\epsilon V_\epsilon = \mathbb{R} \setminus \{0\} \neq \mathbb{R} = \bigwedge_\epsilon (U \vee V_\epsilon) .$$

□

So, using the notation of the introduction, the objects of **Alg** can indeed be chosen to be the frames, so we know need to know what happens at the level of the morphisms.

Definition 2. *A* ***morphism of frames*** *from X to Y is defined to be a function $X \xrightarrow{f} Y$ which preserves the finite infima and all suprema. The category of frames with these morphisms is denoted by* **Frm**.

Since $a \leq b \iff a \vee b = b \iff a \wedge b = a$, a morphism of frames is automatically a morphism of posets and it preserves the top and bottom element because $\bigvee \emptyset = 0, \bigwedge \emptyset = 1$.
Since infinite infima does in general not behave so well (w.r.t. distributivity), one can also expect that frame morphisms does not necessarily preserve arbitrary infima which is indeed the case:

Example 2. Let $f \in \mathbf{Top}(S, T)$, consider

$$f^{-1} : \mathcal{O}(T) \to \mathcal{O}(S) : U \mapsto f^{-1}(U).$$

That this is a morphism of frames follows because the inverse image f^{-1} preserves unions and (finite) intersections.
Let τ_E be the euclidean topology on $\mathbb{R}$ and let τ be the coarsest topology containing τ_E and $\{0\}$. Consider

$$f = Id : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_E).$$

Since τ contains τ_E, this function is continuous. Let $U_\epsilon := (-\epsilon, \epsilon)$ for $\epsilon > 0$. Then

$$\bigwedge f^{-1}(U_\epsilon) = \bigwedge U_\epsilon = \{0\} \neq \emptyset = f^{-1}(\emptyset) = f^{-1}(\bigwedge U_\epsilon).$$

□

In the previous example, we assign to a continuous function $S \xrightarrow{f} T$, a morphism of frames $\mathcal{O}(T) \xrightarrow{f^{-1}} \mathcal{O}(S)$. It is not possible to define a *natural* frame morphism of the form $\mathcal{O}(S) \to \mathcal{O}(T)$. A first reason is that f is in general not open, so $U \mapsto f(U)$ is not well-defined. A way around this would be to consider f^{-1} as a poset morphism which then has a right adjoint given by the following lemma, but it turns out that this adjoint is not a morphism of frames.

Lemma 1. *A frame morphism $\phi : B \to A$ (considered as a poset morphism) has a right adjoint $\psi : A \to B$.*

Proof. By the right adjoint ψ of ϕ, we mean the following: Consider A and B as poset categories, ϕ then corresponds with a functor and ψ is then the right adjoint of this functor. So the adjoint property

$$A(\phi(b), a) \cong B(b, \psi(a)),$$

means precisely

$$\phi(b) \leq a \iff b \leq \psi(a). \tag{1}$$

Define

$$\psi : A \to B : a \mapsto \bigvee \{b \in B | \phi(b) \leq a\}.$$

Then ψ clearly satisfies (1) and if $a_1 \leq a_2$, then $\psi(a_1) \leq \psi(a_2)$, which is the condition for being a poset morphism. □

Note that, because ψ is a right adjoint, it preserves all limits, thus in particular all infima, but in general it does not preserve suprema which shows that ψ is not always a morphism of frames as proven in the following example:

Example 3. *Let $f \in \mathbf{Top}(S, T)$. The right adjoint of f^{-1} (defined previously) considered as a poset morphism, is given by:*

$$f_\star : \mathcal{O}(S) \to \mathcal{O}(T) : U \mapsto \bigcup \{V \in \mathcal{O}(T) | f^{-1}(V) \subseteq U\}.$$

Let $S = T = \{0, 1\}$. Let $\tau_d := \mathbb{P}(S)$ be the discrete topology on S and $\tau_s = \{\emptyset, \{1\}, \{0, 1\}\}$ the sierpinski topology. Consider

$$(\{0, 1\}, \tau_d) \xrightarrow{f=Id} (\{0, 1\}, \tau_s).$$

Since τ_s is coarser then τ_d, it is continuous. We now claim that $f_\star = Id_\star$ does not preserve suprema, indeed: For $U \in \tau_d$ we have:

$$Id_\star(U) = \bigcup \{V \in \tau_s | V \subseteq U\}.$$

So

$$\begin{aligned} Id_\star(\{0\}) &= \emptyset \\ Id_\star(\{1\}) &= \{1\} \\ Id_\star(\{0, 1\}) &= \{0, 1\} \end{aligned}$$

implies

$$Id_\star(\{0\}) \cup Id_\star(\{1\}) = \{1\} \neq \{0, 1\} = Id_\star(\{0, 1\}).$$

which shows that $f_\star$ does not preserves suprema and hence is not a morphism of frames.

So we conclude that instead of working in the category **Frm** of frames (with morphism of frames), we want to work in its opposite category **Locale**, called the category of locales. So a frame is considered as an *algebraic poset* while a locale will be thought of as a *geometrical poset* or as a *generalized space*.
By definition of the opposite category, locales are the same as frames, but the morphisms are different. Another category which has the same objects but different objects is **cHa**, the category of complete Heyting algebras. So we now review those algebras and argue why that is not the wanted category.

Definition 3. *A **Heyting algebra** is a poset which (seen as a category) is finitely complete, finitely cocomplete and cartesian closed. More explicitly: It is a lattice P with top and bottom element such that for each $x, y \in P$, there is some element $(x \implies y) \in P$, called the **exponential**, which is characterized by*

$$\forall z \in P : z \leq (x \implies y) \iff z \wedge x \leq y.$$

Proposition 1. *A lattice is a frame if and only if it is an Heyting algebra.*

Proof. Every frame is a Heyting algebra, one can see this concretely by

$$(x \implies y) := \bigvee \{w | w \wedge x \leq y\}.$$

For the converse: In a lattice, we always have $\bigvee(x \wedge y_i) \leq x \wedge (\bigvee y_i)$. The other inequality follows because:

$$\begin{aligned} x \wedge \left(\bigvee y_i\right) \leq \bigvee (x \wedge y_i) \quad &\iff \quad \bigvee y_i \leq \left(x \implies \bigvee (x \wedge y_i)\right) \\ &\iff \quad \forall i : y_i \leq \left(x \implies \bigvee_j (x \wedge y_j)\right) \\ &\iff \quad \forall i : y_i \wedge x \leq \bigvee_j (x \wedge y_j) \end{aligned}$$

□

This result also follows from the adjoint functor theorem since cartesian closedness can be expressed as saying that $x \wedge - : P \to P$ has a right adjoint.

Definition 4. *A morphism of Heyting algebras is a morphism of frames which preserves exponentials.*

We will show that not each frame morphism preserves exponentials, but first we introduce complementation in a lattice:

Definition 5. *Let L be a lattice with bottom 0 and top 1. A complement for $x \in L$ is some $y \in L$ such that $x \wedge y = 1$ and $x \vee y = 0$.*

In a distributive lattice (so in particular in a frame), a complement is unique and in this case we denote the complement of x by $\neg x$.

Recall that a **boolean algebra** is a distributive algebra with bottom and top in which every element has a complement:

Proposition 2. *Let B be a boolean algebra. Then is B a heyting-algebra by letting $(x \implies y) = \neg x \vee y$.*

One can characterize boolean algebras as certain heyting-algebras. In order to specify this, we define the **negation** of an element x in a Heyting algebra is $\neg x := (x \implies 0)$.

Proposition 3. *In a boolean algebra, the complement and negation coincide and consequently, a lattice is a boolean algebra if and only if it is a heyting algebra and $x \vee \neg x = 1$.*

The following example shows that for $f \in \mathbf{Top}(Y, X)$, the induced frame-morphism f^{-1}, does in general not preserves the negation (and thus the implication), so f^{-1} is not a heyting algebra morphism:

Example 4. Let R have the euclidean topology. For each $U \in \mathcal{O}(\mathbb{R})$ we have $\neg U = int(\mathbb{R} \setminus U)$. We want to show that in general $f^{-1}(\neg U) \neq \neg f^{-1}(U)$. Let

$$0 \equiv f : \mathbb{R} \to \mathbb{R} : x \mapsto 0,$$

be the constant 0-function and let $U := (-\infty, 0)$. Thus $\neg U = int([0, \infty)) = (0, \infty)$, so $f^{-1}(\neg U) = \emptyset$ since $0 \notin (0, \infty)$.
Since $0 \notin (-\infty, 0)$, we have $f^{-1}(U) = \emptyset$, consequently:

$$\neg f^{-1}(U) = int(\mathbb{R} \setminus f^{-1}(U)) = int(\mathbb{R} \setminus \emptyset) = \mathbb{R}.$$

□

3 Locales

Motivated by the previous section, we define the category **Locale** of locales to be the opposite of the category **Frm** of frames (with morphisms of frames), i.e. $\mathbf{Locale} := \mathbf{Frm}^{op}$.
We denote by

$$\mathcal{O} : \mathbf{Locale} \to \mathbf{Frm},$$

the anti-equivalence functor. We also have the *evident* functor

$$Loc : \mathbf{Top} \to \mathbf{Locale}$$

Both will be used implicitly throughout the text and we use the following convention:

Notation 1. *Let* $\mathbf{Space} \in \{\mathbf{Locale}, \mathbf{Top}\}$*. For* $A \in \mathbf{Space}$ *an object and* $f \in \mathbf{Space}$ *a morphism, we denote by* $\mathcal{O}(A)$ *its corresponding frame and* f^{-1} *the corresponding frame morphism. Usually we denote topological spaces by* S, T *and locales by* X, Y *which will make the distinction and otherwise the reader has to pay close attention.*

Since a locale will be thought of as a generalized space, we will introduce concepts based upon the notion of topological spaces. We first introduce the notion of a point in a locale which then leads to the right adjoint of $Loc : \mathbf{Top} \to \mathbf{Locale}$. Then we will show that this adjunction becomes an equivalence of categories when we restrict to a certain subcategory.
Then we also introduce the notions of sublocales, embeddings and surjections which allows us to give (and show) that we have some nice factorization system on **Locale** and as an application we introduce the notion of open and closed sublocales.

3.1 Points and Sober spaces

Let S be a topological space and $\star$ be the 1-point top. space, i.e. the terminal object in **Top**. Since there is a bijection between S and $Top(\star, S)$ (given by sending $s \in S$ to $\tilde{s} : \star \to S : \star \mapsto s$), we define a point of a locale as follows:

Definition 6. *A **point** of a locale* X *is a morphism (of locales)* $p : \mathbf{1} \to X$*, where* $\mathbf{1}$ *is the terminal object in* **Locale***. We denote by* $Pt(X) := \mathbf{Locale}(\mathbf{1}, X)$ *the set of points of* X*.*

Remark 1. *That* **Locale** *has indeed a terminal object is because* $\{0, 1\}$ *is initial in* **Frm***. So a point* p *of* X *is given (equivalently) by a frame morphism* $p^{-1} : \mathcal{O}(X) \to \{0, 1\}$*.*

Lemma 2. *Let* F *be a frame. A function* $f : F \to \{0, 1\}$ *is a morphism of frames if and only if its kernel* $K := f^{-1}(0)$ *satisfies:*

$$\begin{cases} & 1_F \notin K, \\ x \wedge y \in K & \iff x \in K \text{ or } y \in K, \\ \bigvee x_i \in K & \iff \forall i : x_i \in K. \end{cases} \tag{2}$$

Proof. These 3 properties are equivalent to

$$\begin{cases} & f(1_F) = 1, \\ f(x \wedge y) = 0 & \iff f(x) \wedge f(y) = 0 \\ f(\bigvee x_i) = 0 & \iff \forall i : f(x_i) = 0. \end{cases}$$

The first (resp. second) equation is equivalent to $f(x \wedge y) = f(x) \wedge f(y)$ (resp. $f(\bigvee x_i) = \bigvee f(x_i)$) since $cod(f) = \{0, 1\}$. Notice that the condition $f(1_F) = 1$ is necessary to force $f(\bigwedge \emptyset) = 1$. □

Since a function $F \to \{0, 1\}$ is completely defined by the set $f^{-1}(0)$, we conclude that a point can equivalently be defined as a subset K which satisfies (2).

Lemma 3. *Let $F \in \mathbf{Frm}$. There is a bijection between the subsets $K \subseteq F$ satisfying (2) and the proper prime elements, i.e. $p \in F$ such that*

$$\begin{cases} 1_F \neq p, \\ x \wedge y \leq p \iff x \leq p \text{ or } y \leq p. \end{cases}$$

Proof. The bijection is given by $K \mapsto \bigvee K, p \mapsto \downarrow \{p\}$.
Assume $K \subseteq F$ satisfy (2) and let $p := \bigvee K$. We claim that $K = \downarrow \{p\}$, indeed: since K is closed under all suprema we have $p = \bigvee K \in K$ and if $x \leq p$ then is $p = p \vee x$ from which it follows that $x \in K$.
Thus $x \in K \iff x \leq p$ and in particular

$$x \wedge y \leq p \iff x \wedge y \in K \iff x \in K \text{ or } y \in K \iff x \leq p \text{ or } y \leq p.$$

So p is indeed a prime element and it is proper since $1_F \notin K$
Now assume that $p \in F$ is proper prime. Let $K := \downarrow \{p\}$, i.e. $x \in K \iff x \leq p$. Since p is proper, $1_F \notin K$. That K is closed under binary infima follows from:

$$x \wedge y \in K \iff x \wedge y \leq p \iff x \leq p \text{ or } y \leq p \iff x \in K \text{ or } y \in K.$$

That K is closed under arbitrary suprema follows from:

$$\bigvee x_i \in K \iff \bigvee x_i \leq p \iff \forall i : x_i \leq p \iff \forall i : x_i \in K.$$

That the assignments induce a bijection follows since $K = \downarrow \{\bigvee K\}$. □

So all together, we conclude:

Corollary 1. *There is a bijection between:*

1. *the points p of X,*
2. *subsets $K \subseteq \mathcal{O}(X)$ satisfying*

$$\begin{cases} 1 \notin K, \\ U \wedge V \in K \iff U \in K \text{ or } V \in K, \\ \bigvee U_i \in K \iff \forall i : U_i \in K. \end{cases}$$

3. *proper prime elements $P \in \mathcal{O}(X)$.*

and this bijection is given by the following formulas:

$$P = \bigvee K, \quad K = \downarrow P, \quad K = ker(p^{-1}).$$

For a topological space S, each of its points have a *evident* point of the corresponding locale given in the following (equivalent) ways:

- As a frame morphism $p_s : 1 \to Loc(S)$ defined by

$$(p_s)^{-1} : \mathcal{O}(S) \to \{0,1\} : U \mapsto \begin{cases} 1, & \text{if } s \in U \\ 0, & \text{else} \end{cases}$$

- As the proper prime element $P := S \setminus \{\bar{s}\}$
- As the subset $K := \{U \in \mathcal{O}(S) | s \notin U\}$

A sober topological space is where the points can be recovered from the locale as follows:

Definition 7. *A topological space S is **sober** if*

$$S \to Pt(Loc(S)) : s \mapsto p_s, \tag{3}$$

is a bijection.

Proposition 4. *A topological space is sober if and only if for each $P \in \mathcal{O}(S) \setminus \{S\}$ such that*

$$\forall U, V \in \mathcal{O}(S) : U \cap V \subseteq P \implies U \subseteq P \text{ or } V \subseteq P, \tag{4}$$

there exist a unique $s \in S$ such that $P = S \setminus \overline{\{s\}}$.

Proof. Notice that (4) means that P is prime (actually proper prime since $P \neq S$). And since there is a bijection between the proper prime elements and the points, being proper is equivalent to saying that each proper prime element is associated to a unique point in S. But the proper prime elements associated a point is of the form $S \setminus \overline{\{s\}}$ for some $s \in S$. □

Proposition 5. *A topological space is sober if and only if every nonempty irreducible closed set is the closure of a unique point.*

Proof. This is just translation of the previous characterisation into closed sets since if $F \subseteq S$ is closed, $P := S \setminus F$ is open (and $F \neq \emptyset \iff P \neq S$), (4) is then equivalent to saying (using that $U \subseteq S \setminus F \iff F \subseteq S \setminus U$):

$$\forall U, V \in \mathcal{O}(S) : F \subseteq S \setminus (U \cap V) = (S \setminus U) \cup (S \setminus V) \implies F \subseteq S \setminus U \text{ or } F \subseteq S \setminus V.$$

And then using that U is open if and only if $S \setminus U$ is closed and using that a subset is irreducible if it is a irreducible space w.r.t. the subspace topology, we get precisely that $S \setminus F$ satisfies (4) if and only if F is irreducible. □

3.1.1 Sobriety and separation

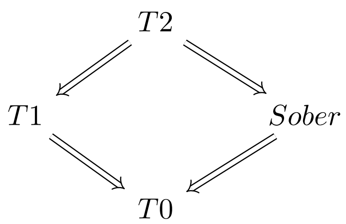
Proposition 6. *Every hausdorff space is sober.*

Proof. Let S be a Hausdorff topological space and assume $F \subseteq S$ is a non-empty, closed and irreducible set with $x \neq y \in F$. By being $T2$, there exists disjoint open neighbourhoods U_x and U_y of x resp. y. Since they are disjoint, $F = (F \setminus U_x) \cup (F \setminus U_y)$. Since U_x and U_y are open in S, $F \setminus U_x = F \setminus (F \cap U_x)$ is closed in F. This contradicts the irreducibility of F. So every non-empty, closed and irreducible set is a singleton (and so is the closure of its unique point). □

Proposition 7. *Every sober space is $T0$.*

Proof. Let S be sober and consider $x \neq y \in S$. Since S is sober, we have that $s \mapsto S \setminus \overline{\{s\}}$ is injective. Thus $\overline{\{x\}} \neq \overline{\{y\}}$, so (without loss of generality), there exists $s \in \overline{\{x\}} \setminus \overline{\{y\}}$. Since $s \notin \overline{\{y\}}$, there exists some open neighbourhood U of s such that $y \notin U$, but since $s \in \overline{\{x\}}$ and $s \in U$, $x \in U$. □

So we have the following diagram:



The following examples show that being sober does not imply $T1$ (and vice versa):

Example 5. *Consider the Sierpinski space, i.e.* $S := \{0,1\}$ *is equipped with the topology* $\{\emptyset, \{1\}, \{0,1\}\}$*. Since* $\{0\}$ *is not open, it is not* $T1$*. But it is sober, indeed:*
Any frame morphism $p : \mathcal{O}(S) \to \{0,1\}$ *must satisfy* $p(\emptyset) = 0, p(S) = 1$*. So* p *is defined by saying what* $p(\{0\})$ *and* $p(\{1\})$ *are. But we also need:*

$$\begin{aligned} 1 = \quad p(\{0,1\}) &= p(\{0\}) \vee p(\{1\}) \\ 0 = \qquad p(\emptyset) &= p(\{0\}) \wedge p(\{1\}) \end{aligned}$$

So $p(\{0\}) \neq p(\{1\})$*, thus there are only 2 choices and if we choose* $p(\{0\}) = 0$ *(so* $p(\{1\}) = 1$*) we get* p_1 *and if we choose* $p(\{0\}) = 1$ *we get* p_0*. So the assignment* $S \to Pt(S) : s \mapsto p_s$ *is indeed a bijection.*

Example 6. *Since any finite space which is* $T1$ *is discrete, it is sober (as such spaces are Hausdorff). So we have to look at infinite spaces. Consider an infinite set* S *with the cofinite topology, i.e.* $U \subseteq S$ *is open if and only if* $S \setminus U$ *is finite. Since* $F \subseteq S$ *is closed if and only if* F *is finite (or* $F = S$*), all points are closed, i.e. it is* $T1$*.*
This space is not sober since S *itself is irreducible (since* S *is infinite and the only infinite closed set is* S *itself) but points are closed so* S *is not the closure of a point.*

3.2 From Locales to Topological spaces

Let X be a locale. For $U \in \mathcal{O}(X)$, set

$$pt(U) := \left\{ p \in Pt(X) | p^{-1}(U) = 1 \right\}.$$

Lemma 4. *The set* $\{pt(U) | U \in \mathcal{O}(X)\}$ *forms a topology on* $Pt(X)$*.*

Proof. Let 1_X (resp. 0_X) be the top (resp. bottom) element of $\mathcal{O}(X)$. Since p^{-1} is a frame morphism (for $p \in Pt(X)$), $p^{-1}(1_X) = 1$ and $p^{-1}(0_X) = 0$, so $Pt(X) = pt(1_X), \emptyset = pt(0_X)$. Thus the whole -and empty set are open.
That the intersection of opens is again open follows because $pt(U \wedge V) = pt(U) \cap pt(V)$, indeed:

$$\begin{aligned} p \in pt(U \wedge V) &\iff p^{-1}(U \wedge V) = 1 \\ &\iff p^{-1}(U) \wedge p^{-1}(V) = 1 \\ &\iff p^{-1}(U) = 1 = p^{-1}(V) \\ &\iff p \in pt(U) \cap pt(V) \end{aligned}$$

That the (arbitrary) intersection of opens is open follows because $pt(\bigvee_i U_i) = \bigcup_i pt(U_i)$, indeed:

$$\begin{aligned} p \in pt(\bigvee_i U_i) &\iff p^{-1}(\bigvee_i U_i) = 1 \\ &\iff \bigvee p^{-1}(U_i) = 1 \\ &\iff \exists i : p \in pt(U_i) \\ &\iff p \in \bigcup_i pt(U_i) \end{aligned}$$

□

Proposition 8. *Let* $f \in$ **Locale**(X, Y)*. The map*

$$Pt(f) : Pt(X) \to Pt(Y) : p \mapsto f \circ p,$$

is continuous for the above topology.

Proof. That $Pt(f)$ is continuous follows because $Pt(f)^{-1}(pt(V)) = pt(f^{-1}(V))$ (for $V \in \mathcal{O}(Y)$), indeed:

$$\begin{aligned} p \in Pt(f)^{-1}(pt(V)) &\iff f \circ p = Pt(f)(p) \in pt(V) \\ &\iff p^{-1}(f^{-1}(V)) = (f \circ p)^{-1}(V) = 1 \\ &\iff p \in pt(f^{-1}(V) \end{aligned}$$

□

Since $Pt(Id_X)(p) = p, Pt(g \circ f)(p) = g \circ f \circ p = Pt(g)(Pt(f)(p))$, we get a (covariant) functor

$$Pt : \mathbf{Locale} \to \mathbf{Top}.$$

Proposition 9. *The functor Pt is the right adjoint of Loc.*

Proof. We have to define isomorphisms

$$\theta = \theta_{S,X} : \mathbf{Top}(S, Pt(X)) \to \mathbf{Locale}(Loc(S), X)$$

which are natural in both S and X. Let $g \in \mathbf{Top}(S, Pt(X))$. Define $\theta(g) = f_g : Loc(S) \to X$ as

$$f_g^{-1} : \mathcal{O}(X) \to S : V \mapsto g^{-1}(pt(V)) = \left\{s \in S | g(s)^{-1}(V) = 1\right\}.$$

Since $V \mapsto pt(V)$ preserves finite infima and arbitrary suprema (since they form a topology) and g^{-1} is also a frame morphism (since g is continuous), f_g^{-1} is a frame morphism, thus $f_g \in \mathbf{Locale}(X, Loc(S))$. Let $f \in \mathbf{Locale}(X, Loc(S))$. For $s \in S$, define $g_f(s) : 1 \to X \in Pt(X)$ as

$$g_f(s)^{-1}(V) = \begin{cases} 1, & \text{if } s \in f^{-1}(V) \\ 0, & \text{else} \end{cases}$$

for $V \in \mathcal{O}(X)$. Define

$$\theta_{S,X}^{-1}(f) = g_f : S \to Pt(X) : s \mapsto g_f(s).$$

Since

$$g_f^{-1}(pt(V)) = \left\{s \in S | g(s)^{-1}(V) = 1\right\} = f^{-1}(V),$$

and f is continuous, we have that g_f is continuous, i.e. $g_f \in \mathbf{Top}(S, Pt(X))$.
That $f \mapsto g_f, g \mapsto f_g$ gives a bijection $\mathbf{Top}(S, Pt(X)) \leftrightarrow \mathbf{Locale}(Loc(S), X)$ follows from:

$$g_{f_g}(s)^{-1}(V) = 1 \iff s \in f_g^{-1}(V) \iff g(s)^{-1}(V) = 1,$$
$$f_{g_f}^{-1}(V) = \left\{s \in S | g_f(s)^{-1}(V) = 1\right\} = \left\{s \in S | s \in f^{-1}(V)\right\} = f^{-1}(V).$$

It remains to show the naturality. We first show that $\theta_{S,X}$ is natural in S. Consider $\phi \in \mathbf{Top}(S_1, S_2)$, we have to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Top}(S_2, Pt(X)) & \xrightarrow{\theta_{S_2,X}} & \mathbf{Locale}(Loc(S_2), X) \\ \downarrow{\scriptstyle -\circ\phi} & & \downarrow{\scriptstyle -\circ Loc(\phi)} \\ \mathbf{Top}(S_1, Pt(X)) & \xrightarrow{\theta_{S_1,X}} & \mathbf{Locale}(Loc(S_1), X) \end{array}$$

The *path below* is defined as (more precisely: the corresponding frame morphism):

$$\theta_{S,X}(f \circ \phi)^{-1} : \mathcal{O}(X) \to \mathcal{O}(S_1) : V \mapsto \left\{s \in S_1 | (f \circ \phi)(s)^{-1}(V) = 1\right\}.$$

The *path above* is given by

$$\begin{aligned} (\theta_{S_2,X}(f) \circ Loc(\phi))^{-1} = \phi^{-1} \circ \theta_{S_2,X}(f)^{-1} : \quad \mathcal{O}(X) &\to \quad \mathcal{O}(S_1) : \\ V &\mapsto \quad \phi^{-1}(\theta_{S_2,X}(f)^{-1}(V)) \end{aligned}$$

And we have

$$\phi^{-1}(\theta_{S_2,X}(f)^{-1}(V)) = \phi^{-1}\left(\left\{s_2 \in S_2 | f(s_2)^{-1}(V) = 1\right\}\right) = \left\{s_1 \in S_1 | f(\phi(s_1))^{-1}(V) = 1\right\}.$$

So we indeed have that the diagram commutes.
To show that $\theta_{S,X}$ is natural in X, we have to show that the following diagram commutes for $\phi \in \mathbf{Locale}(X_1, X_2)$:

$$\begin{array}{ccc} \mathbf{Top}(S, Pt(X_1)) & \xrightarrow{\theta_{S,X_1}} & \mathbf{Locale}(Loc(S), X_1) \\ \downarrow{\scriptstyle Pt(\phi)\circ -} & & \downarrow{\scriptstyle \phi\circ -} \\ \mathbf{Top}(S, Pt(X_2)) & \xrightarrow{\theta_{S,X_2}} & \mathbf{Locale}(Loc(S), X_2) \end{array}$$

The path above is given by:

$$\begin{aligned} (\phi \circ \theta_{S,X_1}(f))^{-1} = \theta_{S,X_1}(f)^{-1} \circ \phi^{-1} : \quad \mathcal{O}(X_2) \to \quad & \mathcal{O}(S) : \\ V \mapsto \quad & \theta_{S,X_1}(f)^{-1}(\phi^{-1}(V)) = \left\{s \in S | f(s)^{-1}(\phi^{-1}(V)) = 1\right\}. \end{aligned}$$

The path below is given by:

$$\begin{aligned} \theta_{S,X_2}(Pt(\phi) \circ f)^{-1} : \quad \mathcal{O}(X_2) \to \quad & \mathcal{O}(S) : \\ V \mapsto \quad & \left\{s \in S | (Pt(\phi) \circ f)(s)^{-1}(V) = 1\right\} \end{aligned}$$

And we have:

$$(Pt(\phi) \circ f)(s)^{-1}(V) = (\phi \circ f^{-1}(s))^{-1}(V) = f(s)^{-1}(\phi^{-1}(V)).$$

□

We now calculate the unit η and counit ϵ of $Loc \dashv Pt$. Denote by

$$\theta : \mathbf{Top}(S, Pt(X)) \to \mathbf{Locale}(Loc(S), X),$$

the natural isomorphism given in the previous proposition. The unit $\eta : Id_{\mathbf{Locale}} \implies Pt \circ Loc$ and counit $\epsilon : Loc \circ Pt \implies Id_{\mathbf{Top}}$ are given by

$$\eta_S = \theta^{-1}(Id_{Loc(S)}), \quad \epsilon_X = \theta(Id_{Pt(X)}).$$

For the counit we have (by definition of θ):

$$\epsilon_X^{-1}(V) = \left\{p \in Pt(X) | Id(p)^{-1}(V) = 1\right\} = Pt(V), \quad V \in \mathcal{O}(X).$$

The unit is given by a continuous function $S \to Pt(Loc(S))$. Fix $s \in S$, then $\eta_S(s) : 1 \to Loc(S)$ which corresponds with the frame morphism

$$\eta_S(s)^{-1} : \mathcal{O}(S) \to \{0,1\} : V \mapsto \begin{cases} 1, & \text{if } s \in Id_{LocS}^{-1}(V) = V \\ 0, & \text{else} \end{cases}$$

So we get that the unit is precisely given by (3).

Proposition 10. *Let $S \in$ **Top**. The following are equivalent:*

1. *S is sober,*
2. *the unit $\eta : S \to Pt(Loc(S))$ is a homeomorphism,*
3. *there is a homeomorphism $S \cong Pt(X)$ for some locale X.*

Proof. **1** $\implies$ **2**: We know that η is continuous so we only have to show that η is open as η is a bijection by being sober. Let $U \subseteq S$ be open, thus

$$\eta(s) \in pt(U) \iff \eta(s)^{-1}(U) = 1 \iff s \in U.$$

Since η is a bijection, we have $s \in U \iff \eta(s) \in \eta(U)$, thus $\eta(U) = pt(U)$, which shows that η is an open map.
2 $\implies$ **3** is immediate by taking $X = Loc(S)$.
3 $\implies$ **1**: Every open subset in $Pt(X)$ is the form $pt(U)$ with $U \subseteq X$ open. Take $P \in \mathcal{O}(X)$ and assume that $pt(P)$ is proper prime.
We have to show that we can write $pt(P) = pt(X) \setminus \{\bar{\phi}\}$ for a unique point $\phi \in Pt(X)$. That $pt(P)$ is written that way, means precisely:

$$\forall V \subseteq X : pt(V) \subseteq pt(P) \iff \phi \notin pt(V) \iff \phi^{-1}(V) = 0.$$

This defines $\phi^{-1} : \mathcal{O}(X) \to \{0,1\}$ uniquely. It remains to show that this is a frame morphism. That ϕ^{-1} preserves the top element follows from:

$$pt(P) \neq Pt(X) = pt(1_X) \implies pt(1_X) \nsubseteq pt(P) \implies \phi^{-1}(1_X) = 1.$$

That it preserves infima, follows from:

$$\begin{aligned}
\phi^{-1}(U \wedge V) = 0 &\iff pt(U \wedge V) \subseteq pt(P) \\
&\iff pt(U) \cap pt(V) \subseteq pt(P) \\
&\iff pt(U) \subseteq pt(P) \text{ or } pt(V) \subseteq pt(P), \qquad \text{since } P \text{ prime} \\
&\iff \phi^{-1}(U) = 0 \text{ or } \phi^{-1}(V) = 0 \\
&\iff \phi^{-1}(U) \wedge \phi^{-1}(V) = 0.
\end{aligned}$$

That is preserves suprema, follows from:

$$\begin{aligned}
\bigvee \phi^{-1}(U_i) = 0 &\iff \forall i : \phi^{-1}(U_i) = 0 \\
&\iff \forall i : pt(U_i) \subseteq pt(P) \\
&\iff \bigcup pt(U_i) \subseteq pt(P) \\
&\iff pt(\bigvee U_i) \subseteq pt(P) \\
&\iff \phi^{-1}(\bigvee U_i) = 0.
\end{aligned}$$

□

Definition 8. *A locale X has* ***enough points*** *if for all $U, V \in \mathcal{O}(X)$:*

$$pt(U) = pt(V) \implies U = V.$$

The following example shows that not all locales have enough points:

Example 7. *Let $\mathbf{B}$ be a complete boolean algebra. A nonzero element $a \in B$ is called an* ***atom*** *if its downset is trivial, i.e.*

$$\forall b \in B : b \leq a \implies b \in \{0, a\}.$$

There is a bijection between the frame morphisms $\mathbf{B} \to \{0,1\}$ and the atoms of $\mathbf{B}$.

Proof. For $b \in Atoms(\mathbf{B})$, define

$$f_b : \mathbf{B} \to 2 : a \mapsto \begin{cases} 1, & \text{if } a \leq \neg b \\ 0, & \text{else} \end{cases}$$

So this is the morphism/function associated to $K :=\downarrow \{\neg b\}$ in the notation of (2). To show that f_b is a frame morphism, it has to satisfy (2). Clearly we have $1_{\mathbf{B}} \notin K$ since $b \neq 0$. The condition

$$x \wedge y \in K \iff (x \in K \text{ or } y \in K)$$

is equivalent to

$$x \wedge y \leq \neg b \iff (x \leq \neg b \text{ or } y \leq \neg b)\,. \tag{5}$$

First notice that if b is an atom, then

$$b \leq p \vee q \implies (b \leq p \text{ or } b \leq q) \tag{6}$$

Indeed: We have $\{b \wedge p, b \wedge q\} \subseteq \{0, b\}$ (since b is an atom), but we also have

$$b = b \wedge (p \vee q) = (b \wedge p) \vee (b \wedge q).$$

So if both $b \wedge p$ and $b \wedge q$ would be 0, we would have $b = 0$ which is not possible so we indeed have $b \wedge p = b$ or $b \wedge q = b$ which means precisely $b \leq p$ or $b \leq q$.
To show equation (5), we clearly have the implication $\impliedby$ because $x \wedge y \leq x \leq \neg b$, so it remains to show $\implies$. This follows from equation (6):

$$\begin{aligned} f_b(x \wedge y) = 0 \quad &\iff \quad x \wedge y \leq \neg b \\ &\iff \quad b \leq \neg(x \wedge y) = \neg x \vee \neg y \\ &\implies \quad b \leq \neg x \text{ or } b \leq \neg y \\ &\iff \quad x \leq \neg b \text{ or } y \leq \neg b \\ &\iff \quad f_b(x) = 0 \text{ or } f_b(y) = 0 \end{aligned}$$

The last condition to be checked (in order that f_b is a frame morphism) is

$$\bigvee x_i \in K \iff \forall i : x_i \in K,$$

which is equivalent to

$$\bigvee x_i \leq \neg b \iff \forall i : x_i \leq \neg b,$$

which clearly holds, so f_b is indeed a frame morphism.
That the assignment $b \mapsto f_b$ is injective follows from:

$$\begin{aligned} f_a = f_b \quad &\implies \quad \forall x \in B : f_a(x) = 1 \iff f_b(x) = 1 \\ &\implies \quad \forall x \in B : x \leq \neg a \iff x \leq \neg b \\ &\implies \quad a \leq b \text{ and } b \leq a, \qquad \text{by } x = \neg b, x = \neg a \\ &\implies \quad a = b \end{aligned}$$

That the assignment is surjective, let $f \in \mathbf{Frm}(\mathbf{B}, 2)$. Define $K := f^{-1}(\{0\})$ and $b := \neg \bigvee K$. We claim that $f = f_b$, indeed:

$$f_b(a) = 0 \iff a \leq \neg b = \bigvee K = \bigvee f^{-1}(0) \iff f(a) = 0.$$

So it remains to show that b is an atom. Assume $0 < x \leq b$. Thus

$$x \vee (\neg x \wedge b) = (x \vee \neg x) \wedge (x \vee b) = b.$$

So

$$1 = f(b) = f(x) \vee f(\neg x \wedge b).$$

So $f(x) = 1$ or $f(\neg x \wedge b) = 1$. If $\neg x \wedge b \geq b$, then

$$x \leq b \leq b \wedge \neg x \leq \neg x,$$

which is a contradiction (since $x > 0$), so $f(x) = 1$, i.e $x \geq b$. Thus $b = x$, so b is indeed an atom. □

So if **B** *is a complete boolean algebra, it has no points if it is atomless.*
Let $\mathcal{M}$ *be consists of the measurable subsets of* $[0,1]$*. Let* I *be the ideal generated by those Borel sets with measure* 0*. Then is* $\mathcal{M}/I$ *a complete boolean algebra and it has no atoms since no set has measure* 0*. This then gives an example of a locale with no points.*

Proposition 11. *Let* $X \in \mathbf{Locale}$*. The following are equivalent:*

1. *X has enough points,*
2. *the counit* $\epsilon : Loc(Pt(X)) \to X$ *is an isomorphism (of locales)*
3. *there is an isomorphism* $X \cong Loc(S)$ *for some topological space* S*.*

Proof. $\mathbf{1} \implies \mathbf{2}$: (2) is equivalent with saying that $\epsilon{-1}$ is an isomorphism of frames, e.g. we can show that ϵ^{-1} is injective and surjective. Since $\epsilon^{-1}(V) = pt(V)$ and each open in $Pt(X)$ is of the form $pt(V)$ for some $V \in \mathcal{O}(X)$ it is surjective. And that it is injective means precisely that X has enough points, indeed:

$$pt(V) = \epsilon^{-1}(V) = \epsilon^{-1}(U) = pt(U) \implies U = V.$$

$\mathbf{2} \implies \mathbf{3}$: Immediate by $S := Pt(X)$.
$\mathbf{3} \implies \mathbf{1}$: So we have to show that $Loc(S)$ has enough points for all topological spaces S. Assume $U \neq V \in \mathcal{O}(S)$, without loss of generality, let $s \in U \setminus V$. Consider again $p_s : 1 \to Loc(S)$, i.e.

$$p_s^{-1} : \mathcal{O}(S) \to \{0,1\} : W \mapsto \begin{cases} 1, & \text{if } s \in W \\ 0, & \text{else} \end{cases}$$

So $p_s^{-1}(U) = 1$ and $p_s^{-1}(V) = 0$, so by definition of $pt(U)$ (resp. $pt(V)$) we have $p_s \in pt(U)$ (resp. $p_s \notin pt(V)$) which shows the claim. □

Corollary 2. *The adjunction* $Loc \dashv Pt$ *restricts to an equivalence of categories between the (full) subcategories of locales with enough points and sober topological spaces.*

Remark 2. *In the introduction, we have discussed that* $\mathbf{Top} \to \mathbf{Frm} : S \mapsto \mathcal{O}(S)$ *is actually a representable functor with representing object the Sierpinski space* $\mathbf{2}$*. Remarkably,* $Pt : \mathbf{Frm} \to \mathbf{Top} : F \mapsto Pt(\mathcal{O}^{-1}(F))$ *is also representable with the same representing object. We now describe these claims in more detail:*
Consider $S \in \mathbf{Top}$ *and let the set* $\mathbf{2}$ *have the Sierpinski topology with* 1 *as open point. Recall that the bijection from* $\mathcal{O}(S)$ *to* $\mathbf{Top}(S, \mathbf{2})$ *is given by*

$$\mathcal{O}(S) \to \mathbf{Top}(S, \mathbf{2}) : U \mapsto f_U,$$

with

$$f_U : S \to \mathbf{2} : s \mapsto \begin{cases} 1, & \textit{if } s \in U \\ 0, & \textit{else} \end{cases}$$

Let $\mathbf{2}$ *have the evident frame structure, then* $\mathbf{Top}(S, \mathbf{2})$ *has also a frame structure (given pointwise) and this frame structure coincides with the structure on* $\mathcal{O}(S)$ *because:*

$$f_U \leq f_V \iff \forall x \in S : f_U(s) \leq f_V(s) \iff [\forall s \in S : s \in U \implies s \in V] \iff U \subseteq V.$$

Thus we have $\mathcal{O}(S) \cong_{\mathbf{Frm}} \mathbf{Top}(S, \mathbf{2})$*. Let* $f \in \mathbf{Top}(S, T)$*. So we have*

$$\mathbf{Top}(T, \mathbf{2}) \xrightarrow{\mathbf{Top}(f, \mathbf{2})} \mathbf{Top}(S, \mathbf{2}) : f_U \mapsto f_U \circ f.$$

Since

$$f_U \circ f : T \to \mathbf{2} : t \mapsto \begin{cases} 1, & \textit{if } f(t) \in U \\ 0, & \textit{else} \end{cases}$$

we have that $\mathbf{Top}(f, \mathbf{2})$ *represents* f^{-1}*, i.e.*

$$\mathcal{O}(f) = \mathbf{Top}(f, \mathbf{2}).$$

Thus we indeed have that $\mathcal{O}(-)$ *is representable by* $\mathbf{Top}(-, \mathbf{2})$*.*
Now consider $X \in \mathbf{Frm}$*. We have* $Pt(X) = \mathbf{Frm}(X, \mathbf{2})$ *(notice that here we actually used* $Pt(\mathcal{O}^{-1}(X))$ *and the anti-equivalence of frames and locales). Consider again on* $\mathbf{2}$ *the sierpinski topology. On* $\mathbf{Frm}(X, \mathbf{2})$*, we place the* ***compact open topology*** *(we put on* X *the discrete topology), that is:*
The subbase on $\mathbf{Frm}(X, \mathbf{2})$ *consists of the open sets*

$$N(C, U) := \{f \in \mathbf{Frm}(X, \mathbf{2}) | f(C) \subseteq U\}, \quad C \subseteq X \text{ compact}, U \in \mathcal{O}(\mathbf{2}).$$

So

$$\begin{aligned} N(C, \{0,1\}) &= \mathbf{Frm}(X, 2) \\ N(C, \emptyset) &= \begin{cases} \emptyset, & \text{if } C \neq \emptyset \\ \mathbf{Frm}(X, 2), & \text{if } C = \emptyset \end{cases} \\ N(C, \{1\}) &= \bigcap_{c \in C} Pt(c) \end{aligned}$$

So $N(C, \{0,1\})$ *and* $N(C, \emptyset)$ *are open in* $Pt(X)$ *and since* C *is finite (as it is compact in a discrete space),* $N(C, \{1\})$ *is also open in* $Pt(X)$*. Thus the compact open topology is contained in* $Pt(X)$*. Since for each* $U \in \mathcal{O}(X)$*,* $\{U\}$ *is compact in* X*, we have* $Pt(U) = N(\{U\}, \{1\})$ *which shows that the topologies coincides and thus* $Pt(X) =_{\mathbf{Top}} \mathbf{Frm}(X, \mathbf{2})$*.*
For $f \in \mathbf{Frm}(X, Y)$*, we clearly have* $Pt(f) = \mathbf{Frm}(f, \mathbf{2})$ *since* $Pt(f)$ *is also just composition. So we also have* $Pt(-) = \mathbf{Frm}(-, \mathbf{2})$*.*
As a final remark, we notice the importance of $\mathbf{2}$ *as it is the representable object for both Loc and* Pt*, where in both cases we have considered this set as a frame and as a topological space (with the Sierpinski-topology). Objects which give rise to an adjunction in this way are sometimes called* ***schizophrenic****.*

3.3 Embeddings and surjections of Locales

Lemma 5. *Let* $f \in \mathbf{Top}(T, S)$ *and* $f^{-1} \in \mathbf{Frm}(\mathcal{O}(S), \mathcal{O}(T))$ *the corresponding frame morphism.*

- *If* f *is surjective, then is* f^{-1} *injective and the converse hold if* S *is* T_1*.*
- *If* f *is injective, then is* f^{-1} *surjective and the converse hold if* T *is* T_0*.*

Proof. If f is surjective, then for each $U \in \mathcal{O}(T)$, $ff^{-1}(U) = U$, so if $f^{-1}(U) = f^{-1}(V)$:

$$U = ff^{-1}(U) = ff^{-1}(V) = V,$$

which shows that f^{-1} is injective.
Assume f^{-1} is injective and let $s \in S$. If S is T_1, then is $S \setminus \{s\}$ open. Since f^{-1} is injective, $f^{-1}(S \setminus \{s\}) \neq f^{-1}(S) = T$ from which it follows that there is some $t \in T$ with $f(t) = s$ which shows that f is surjective when S is T_1.
Assume $f : T \to S$ is injective, thus T is a subspace of S (considered with the subspace topology). So without loss of generality, $T \subseteq S$ and $\mathcal{O}(T) = \{U \wedge T | U \in \mathcal{O}(S)\}$ and f is the inclusion of $T \subseteq S$ and so $f^{-1}(U) = U \cap T$ which show that f^{-1} is surjective since each open in T is of the form $U \cap T$ with $U \in \mathcal{O}(S)$.
Assume f^{-1} is surjective and let $t \neq \tilde{t}$. If T is T_0, there exists an open $U \in \mathcal{O}(T)$ with $t \in U$ and $\tilde{t} \notin U$. Since f^{-1} is surjective, there is some $V \in \mathcal{O}(S)$ with $f^{-1}(V) = U$, so $f(t) \in V$ and $f(\tilde{t}) \notin V$, so we necessarily have $f(t) \neq f(\tilde{t})$ which shows that f is injective if T is T_0. □

As motivated by the previous lemma, we define:

Definition 9. *A morphism* $f \in \mathbf{Locale}(T, S)$ *of locales is* ***an embedding*** *(resp.* ***a surjection****) if* $f^{-1} : \mathcal{O}(S) \to \mathcal{O}(T)$ *is surjective (resp. injective).*

So the lemma means precisely:

Corollary 3. *Let $f \in \mathbf{Top}(T, S)$. Then*

- *If S is T_1, f is surjective if and only if $Loc(f)$ is a surjection.*
- *If T is T_0, f is injective if and only if $Loc(f)$ is an embedding.*

Recall that each frame morphism (considered as a morphism of posets) has a right adjoint (1).

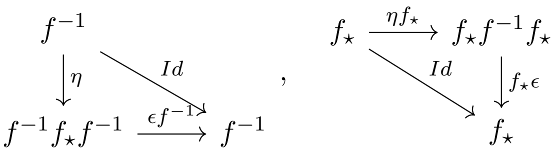
Lemma 6. *Let $f \in \mathbf{Locale}(Y, X)$. Let $f_\star : \mathcal{O}(Y) \to \mathcal{O}(X)$ be the right adjoint (of posets) of $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(Y)$. Then*

1. $\forall U \in \mathcal{O}(X) : U \leq f_\star f^{-1} U$,
2. $\forall V \in \mathcal{O}(Y) : f^{-1} f_\star V \leq V$,
3. $f^{-1} f_\star f^{-1} = f^{-1}$,
4. $f_\star f^{-1} f_\star = f_\star$.

Proof. The first 2 properties follows by the existence of the unit and counit of the adjunction, more precisely it follows from:

$$\mathcal{O}(X)(U, f_\star f^{-1}U) \cong \mathcal{O}(X)(f^{-1}U, f^{-1}U), \quad \mathcal{O}(Y)(f^{-1}f_\star V, V) \cong \mathcal{O}(Y)(f_\star V, f_\star V).$$

The other 2 properties are *visualized* by the triangular identities of the adjunction in the following way: The triangular identities tells us that the following diagrams commute:



So the left diagram tells us that for each $U \in \mathcal{O}(X)$ there are frame morphisms

$$f^{-1}(U) \xrightarrow{\eta_U} f^{-1}f_\star f^{-1}(U), \quad f^{-1}f_\star f^{-1}(U) \xrightarrow{\epsilon f^{-1}(U)} f^{-1}(U)$$

and by definition of a poset category this means precisely that

$$f^{-1}f_\star f^{-1}(U) \leq f^{-1}(U), \quad f^{-1}(U) \leq f^{-1}f_\star f^{-1}(U),$$

so we conclude $f^{-1} = f^{-1}f_\star f^{-1}$ which shows (3). In the exact same way, (4) follows from the right diagram. ☐

An immediate consequence of the lemma is:

Proposition 12. *Let $f \in \mathbf{Locale}(Y, X)$. The following are equivalent:*

1. *f is a surjection.*
2. *$f_\star f^{-1} = Id : \mathcal{O}(X) \to \mathcal{O}(X)$.*
3. *$f_\star : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a surjection of posets.*

And dually, the following are equivalent:

1. *f is an embedding.*
2. *$f^{-1} f_\star = Id : \mathcal{O}(Y) \to \mathcal{O}(Y)$.*
3. *$f_\star : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective.*

Proof. From $f^{-1}f_\star f^{-1}U = f^{-1}U$ it follows that if f^{-1} is injective, then is $f \star f^{-1} = Id_{\mathcal{O}(X)}$ (so (1) $\implies$ (2)).
If (2) holds, i.e. $U = f_\star f^{-1}U$, we immediately have that $f_\star$ is surjective (so (2) implies (3)).
Assume $f^{-1}(U_1) = f^{-1}(U_2)$, if $f_\star$ is surjective, we have $U_i = f_\star V_i$ for some $V_i \in \mathcal{O}(Y)$. Injectivity of f^{-1} then follows from the triangular identity:

$$U_1 = f_\star V_1 = f_\star f^{-1} f_\star V_1 = f_\star f^{-1} f_\star V_2 = f_\star V_2 = U_2.$$

□

3.3.1 Sublocales

Definition 10. *Let $X \in$ **Locale**. A **nucleus** on X is a function*

$$j : \mathcal{O}(X) \to \mathcal{O}(X),$$

such that:

- $U \leq jU$
- $j^2U = jU$
- $j(U \wedge V) = jU \wedge jV$

Proposition 13. *If $f \in$ **Locale**(Y, X) is an embedding, then is $j := f_\star \circ f^{-1}$ a nucleus on X. And $\mathcal{O}(Y)$ is isomorphic to*

$$\{U \in \mathcal{O}(X) | jU = U\}.$$

Proof. We already know $U \leq f_\star f^{-1}U = jU$, thus in particular $jU \leq j^2U$ (since $f_\star, f^{-1}$ are morphisms of posets and respect the ordering). We also claim that $j^2(U) \leq j(U)$, indeed: Since every adjunction induces a monad, its multiplication gives the desired inequality, more precisely: Since $f_\star \vdash f^{-1}$, there is some associated monad (j, ϵ, η) on $\mathcal{O}(X)$ where $\eta : j^2 \to j$ is a natural transformation (the *multiplication*), so for each $U \in \mathcal{O}(X)$ we have a morphism $\eta_U : j^2(U) \to j(U)$ which means $j^2(U) \leq j(U)$. So we conclude $j = j^2$. The last property to check is that j preserves finite infima which follows since $f_\star$ (as it is a right adjoint) and f^{-1} (as it is a frame morphism) preserves finite infima. Thus j is indeed a nucleus on X.
Since $f_\star$ is injective, we have

$$\mathcal{O}(Y) = f_\star(\mathcal{O}(Y)) = \{f_\star(U) | U \in \mathcal{O}(Y)\}.$$

Using the triangular identity we have $f_\star U = f_\star f^{-1} f_\star U$, but $f_\star$ is injective, thus $U = f^{-1} f_\star U$. From this it follows that

$$\mathcal{O}(Y) \cong \{U \in \mathcal{O}(X) | jU = U\}.$$

□

Proposition 14. *If j is a nucleus on X, the set of fixed points of j*

$$F := \{U \in \mathcal{O}(X) | jU = U\},$$

is a frame and j becomes a surjective frame morphism into this frame.

Proof. That F is closed under infima follows because j preserves infima and thus the infima in F is the infima in $\mathcal{O}(X)$.
We claim that the suprema of a set $\{U_\alpha\} \subseteq F$ is $j\left(\bigvee_\alpha^{\mathcal{O}(X)} U_\alpha\right)$, indeed:

- Since $U_\alpha \leq \bigvee_\alpha^{\mathcal{O}(X)} U_\alpha$, we have (since j is a nucleus) $j(U_\alpha) \leq j\left(\bigvee_\alpha^{\mathcal{O}(X)} U_\alpha\right)$.

- Let $V \in F$ such that $U_\alpha \leq V$ (forall α), since $V \in \mathcal{O}(X)$, we have $\bigvee^{\mathcal{O}(X)} U_\alpha \leq V$ which implies (since j is a nucleus and $V \in F$)
$$j\left(\bigvee^{\mathcal{O}(X)} U_\alpha\right) \leq j(V) = V.$$

Since j preserves $\leq$, we clearly have that $j(\emptyset)$ (resp. $j(X)$) is the bottom (resp. top) element of F. So F is indeed a frame and the image of j is indeed F because $j^2 = j$ which shows $j(\mathcal{O}(X)) \subseteq F$ and the other inclusion is immediate since $jU = U$ which shows that j is surjective onto F. □

Definition 11. *Let j be a nucleus on X. The locale corresponding to the frame of fixed points of j is denoted by X_j, i.e.*
$$\mathcal{O}(X_j) := \{U \in \mathcal{O}(X) | jU = U\}.$$
*Locales of the form X_j are called **sublocales** of X.*

The previous proposition becomes:

Corollary 4. *Let X_j be a sublocale of X (as in the above definition). Then determines j an embedding of locales $i : X_j \to X$ by $i^{-1}(U) := jU$.*

3.3.2 Factorizations in Locale

We are going going to show that each morphism of locales factorizes (uniquely) through a sublocale, but we first need the following lemma:

Lemma 7. ***("Factorization lemma")*** *Let $f \in$ **Locale**(Y, X) and j a nucleus on X with the embedding $i : X_j \to X$. Then f factors (uniquely) through i if and only if $f^{-1} \circ j = f^{-1}$.*

Proof. Assume $f = i \circ p$ for some $p : Y \to X_j$. Since $i^{-1} = j$, we get
$$f^{-1}(U) = p^{-1} \circ j(U) = p^{-1} \circ j^2(U) = f^{-1} \circ j(U).$$
Conversely assume $f^{-1} \circ j = f^{-1}$. Define
$$p^{-1} : \mathcal{O}(X_j) \to \mathcal{O}(Y) : U \mapsto f^{-1}(U).$$
Since the infima in $\mathcal{O}(X_j)$ is the same as in $\mathcal{O}(X)$, p^{-1} preserves infima since f preserves infima, more precisely:
$$\begin{aligned} p^{-1}(U) \wedge_{\mathcal{O}(Y)} p^{-1}(V) &= f^{-1}(U) \wedge_{\mathcal{O}(Y)} f^{-1}(V) \\ &= f^{-1}\left(U \wedge_{\mathcal{O}(X)} V\right) \\ &= f^{-1}\left(U \wedge_{\mathcal{O}(X_j)} V\right) \\ &= p^{-1}\left(U \wedge_{\mathcal{O}(X_j)} V\right). \end{aligned}$$
Since the suprema in $\mathcal{O}(X_j)$ is computed as the suprema in $\mathcal{O}(X)$ followed by applying j and $f^{-1} \circ j = f^{-1}$,

p^{-1} preserves suprema, more precisely:

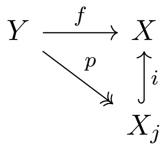
$$\begin{aligned}
p^{-1}\left(\bigvee_i^{\mathcal{O}(X_j)} U_i\right) &= p^{-1}\left(j\left(\bigvee_i^{\mathcal{O}(X)} U_i\right)\right) \\
&= f^{-1}\left(j\left(\bigvee_i^{\mathcal{O}(X)} U_i\right)\right) \\
&= f^{-1}\left(\bigvee_i^{\mathcal{O}(X)} U_i\right) \\
&= \bigvee_i^{\mathcal{O}(Y)} f^{-1}(U_i) \\
&= \bigvee_i^{\mathcal{O}(Y)} p^{-1}(U_i)
\end{aligned}$$

So p^{-1} is indeed a morphism of frames and thus defines a morphism of locales $p : Y \to X_j$. That $f = i \circ p$ follows from:

$$p^{-1}i^{-1}U = f^{-1}jU = f^{-1}U.$$

Since i^{-1} is a surjection, such a p is unique. □

Theorem 1. ***(Factorization theorem, existence)*** *Let $f \in$ **Locale**(Y, X). There exists a nucleus j on X such that f factors through the embedding $i : X_j \to X$ via a surjection p, i.e.*



Proof. Let $j := f_\star f^{-1}$, by (proposition 13) we know this is a nucleus on X. From the triangular identity of $f^{-1} \dashv f_\star$, it follows from the previous lemma that there exists a factorization p of f through j, more precisely it follows from:

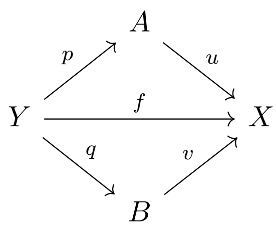
$$f^{-1}j = f^{-1}f_\star f^{-1} = f^{-1}.$$

For p to be a surjection of locales, p^{-1} has to be injective which indeed is the case because if $p^{-1}U = p^{-1}V$, we have by definition $f^{-1}U = f^{-1}V$, thus $f_\star f^{-1}U = f_\star f^{-1}V$, but since U and V are in $\mathcal{O}(X_j)$, we have $f_\star f^{-1}U = U$ (resp. for V) which shows $U = V$. □

Corollary 5. *Every embedding of locales is of the form $X_j \to X$.*

Proof. Let $f : Y \to X$ be an embedding. Factorize f as $f = p \circ i$ as above. Since f is an embedding, we have that p is also an embedding, but since p is already a surjection, we have that p isomorphism. □

Theorem 2. ***(Factorization theorem, uniqueness)*** *Let $f \in$ **Locale**(Y, X). Assume f has two factorizations:*



If v is an embedding and p a surjection, then there exists a unique $g \in \mathbf{Locale}(A, B)$ such that $gp = q$ and $vg = u$. If moreover u is also an embedding and q a surjection, this g is an isomorphism.

Proof. Using the corollary, we can assume that $B = X_j$ and $v^{-1} = j$ for some nucleus j on X.
Since f factors through v and $v^{-1} = j$, the factorization lemma tells us that $f^{-1}j = f^{-1}$, so (using $f = up$) we have $p^{-1}u^{-1}j = p^{-1}u^{-1}$, but p^{-1} is injective, thus $u^{-1}j = u^{-1}$. So by applying the lemma again, there exists some locale morphism g such that $u = vg$ which shows the first part of the theorem.
If u is an embedding, then so is g (since $u = vg$) and if q is a surjection, then so is g (since $q = gp$) and since a embedding which is a surjection is an isomorphism, the second part is proven. □

3.3.3 Open sublocales

Let $X \in \mathbf{Locale}$ and $U \in \mathcal{O}(X)$. Then is

$$\downarrow U = \{V \in \mathcal{O}(X) | V \leq U\},$$

(clearly) a frame, so we get a corresponding locale $\underline{U}$, i.e. $\mathcal{O}(\underline{U}) = \downarrow U$.
We also have an evident surjective frame morphism

$$\mathcal{O}(X) \xrightarrow{U \wedge -} \downarrow U : V \mapsto V \wedge U,$$

which corresponds with some embedding of locales

$$f : \underline{U} \hookrightarrow X,$$

i.e. $f^{-1} = U \wedge -$.
The right adjoint of f^{-1} is given by

$$f_\star : \downarrow U \to \mathcal{O}(X) : V \mapsto \bigvee \{W \in \mathcal{O}(X) | W \wedge U = f^{-1}W \leq V\}.$$

So $f_\star(V) = (U \implies V)$. Since f is an embedding, we have that $\underline{U} \cong X_j$ with $j = f_\star f^{-1}$ and

$$\begin{aligned} \forall W \in \mathcal{O}(X) : j(W) &= (U \implies (U \wedge W)) \\ &= \bigcup \{V \in \mathcal{O}(X) | V \wedge U \leq W\} \\ &= \bigcup \{V \in \mathcal{O}(X) | V \wedge U \leq W \wedge U\} \\ &= (U \implies W). \end{aligned}$$

Definition 12. *A sublocale $X_j \to X$ is **open** if there is some $U \in \mathcal{O}(X)$ such that $j(-) = (U \implies -)$.*

3.3.4 Closed sublocales

Let $X \in \mathbf{Locale}$ and $U \in \mathcal{O}(X)$. Then is

$$\uparrow U = \{V \in \mathcal{O}(X) | U \leq V\},$$

(clearly) a frame, so we get a corresponding locale $\underline{X \setminus U}$, i.e. $\mathcal{O}(\underline{X \setminus U}) = \uparrow U$.
We also have an evident frame morphism

$$\mathcal{O}(X) \xrightarrow{U \vee -} \uparrow U : V \mapsto V \vee U,$$

which corresponds with some morphism of locales

$$g : \underline{X \setminus U} \hookrightarrow X,$$

i.e. $g^{-1} = U \vee -$.
The right adjoint of g^{-1} is given by

$$g_\star : \uparrow U \to \mathcal{O}(X) : V \mapsto \bigvee \{W \in \mathcal{O}(X) | W \vee U = g^{-1}W \leq V\}.$$

Since g is an embedding, we have that $\underline{X \setminus U} \cong X_j$ with $j = g_\star g^{-1}$ and

$$\forall W \in \mathcal{O}(X) : j(W) = g_\star g^{-1}(W) = \bigvee \left\{ V \in \mathcal{O}(X) | V \vee U = g^{-1}(V) \leq g^{-1}(W) = W \vee U \right\}.$$

We now claim that $g_\star g^{-1}(W) = W \vee U$. We clearly have $W \vee U \leq g_\star g^{-1} W$ and the other inequality follows from $V \leq V \vee U \leq W \vee U$.

Definition 13. *A sublocale $X_j \to X$ is* ***closed*** *if there is some $U \in \mathcal{O}(X)$ such that $j(-) = (U \vee -)$.*

3.3.5 Sublocales of locales with enough points

Let $X \in$ **Locale** have enough points, i.e. $X = Loc(T)$ with $T \in$ **Top**. If $S \subseteq T$ is a subspace, then the inclusion $i : S \hookrightarrow T$ induces an embedding $Loc(i) : Loc(S) \to Loc(T)$ (coming from $Loc(i)^{-1} = i^{-1}$) which makes $Loc(S)$ into a sublocale of $Loc(T)$. But not every sublocale of $Loc(T)$ will be of the form $Loc(S)$:

Proof. Let T be Hausdorff with no isolated points (so no singletons are open). Since $\mathcal{O}(T)$ is a frame, it is a Heyting algebra and thus we have the negation operator $\neg$. The double negation, that is $\neg \circ lnot$ is a nucleus on $Loc(T)$, so we have a sublocale $Loc(T)_{\neg\neg}$ of $Loc(T)$. We now claim that this sublocale doesn't have any points which shows that $Loc(T)_{\neg\neg}$ doesn't have enough points. Let $p : \mathbf{1} \to Loc(T)_{\neg\neg}$ be a point. Denote by $u : Loc(T)_{\neg\neg} \to Loc(T)$ be the embedding, so $u^{-1}(W) = \neg\neg W$. Thus $u \circ p$ is a point of $Loc(T)$. Since T is hausdorf, it is sober, thus that point is form

$$(up)^{-1} : \mathcal{O}(T) \to \{0, 1\} : W \mapsto \begin{cases} 1, & \text{if } t \in W \\ 0, & \text{else} \end{cases}$$

for a unique $t \in T$. Thus

$$0 = (up)^{-1}(T \setminus \{t\}) = p^{-1} \circ u^{-1}(T \setminus \{t\}) = p^{-1}(\neg\neg(T \setminus \{t\})). \tag{7}$$

But t is not isolated, thus

$$\neg(T \setminus \{t\}) = int(T \setminus (T \setminus \{t\})) = int(\{t\}) = \emptyset.$$

Applying $\neg$ again we thus get $\neg\neg(T \setminus \{t\}) = \neg(\emptyset) = int(T \setminus \emptyset) = T$. Since $t \in T$, we thus have

$$p^{-1}(\neg\neg(T \setminus \{t\})) = p^{-1}(T) = 1,$$

which contradicts (7). □

4 Localic topoi

Given a frame F, one can define a base for a grothendieck topology by

$$\{U_i \to U\}_{i \in I} \text{ covers } U \iff \bigvee_{i \in I} U_i = U.$$

So this induces a grothendieck topos $Sh(F)$:

Definition 14. *A* ***localic topos*** *is a category equivalent to $Sh(F)$, or equivalently a category equivalent to $Sh(\mathcal{O}(X))$ where X is a locale. If X is a locale, we write $Sh(X) := Sh(\mathcal{O}(X))$.*

Let $\mathcal{E}$ and $\mathcal{F}$ be (elementary) topoi. Recall that a *geometric morphism* $f : \mathcal{E} \to \mathcal{F}$ is given by a pair of functors

$$\mathcal{E} \underset{f^\star}{\overset{f_\star}{\rightleftarrows}} \mathcal{F}$$

such that $f^\star$ is left adjoint to $f_\star$ and $f^\star$ is left exact (i.e. preserves finite limits). The category of (elementary) topoi with the geometric morphisms as morphisms is denoted by **ETopos** and we denote by **Topos** its full subcategory of grothendieck topoi.

4.1 Subobjects in topoi

Lemma 8. *Let $\mathcal{E}$ be a topos and $A \in \mathcal{E}$ an object. Then is $Sub_{\mathcal{E}}(A)$ a Heyting algebra. If moreover, $\mathcal{E}$ is either complete of cocomplete, then it is a complete Heyting algebra or equivalently a frame.*

Proof. Notice that a subobject is an equivalence of monomorphisms, but we will fix a representative. It is a routine computation which will show that up to equivalence everything works well. The top element is A itself (more precisely Id_A). The bottom element is the initial sheaf $\mathbf{0}$.
Let $S, T \in Sub_{\mathcal{E}}(A)$. The infima is given by the pullback of $S \hookrightarrow A$ along $T \hookrightarrow A$, i.e. the following diagram is a pullback square:

$$\begin{array}{ccc} S \wedge T & \longrightarrow & T \\ \downarrow & & \downarrow \\ S & \longrightarrow & A \end{array}$$

To construct the suprema, consider the coproduct $S \amalg T$ of S and T. In general, the natural morphism $S \amalg T \to A$ is not a monomorphism, so we take its image factorisation

$$\begin{array}{ccccc} & & S & & \\ & \swarrow & & \searrow & \\ S \amalg T & \twoheadrightarrow & I & \hookrightarrow & A \\ & \nwarrow & & \nearrow & \\ & & T & & \end{array}$$

The image factorisation always exists in a (elementary) topos[1]. The suprema of S and T is then given by the monomorphism $I \to A$. This construction given the structure of a lattice on $Sub(A)$.
To show that it has the structure of a Heyting algebra, one uses the isomorphism $Sub_{\mathcal{E}}(A) \cong Sub_{\mathcal{E}/A}(\mathbf{1})$ to assume without loss of generality that we can take A to be the initial object $\mathbf{1}$ in $\mathcal{E}$. The exponential of $S \hookrightarrow \mathbf{1}$ with $T \hookrightarrow \mathbf{1}$ is then given by the unique morphism $S^T \hookrightarrow \mathbf{1}$ where S^T is the exponential given by the cartesian closedness of $\mathcal{E}$.
If $\mathcal{E}$ is cocomplete, all suprema exists by construction. If $\mathcal{E}$ is complete, it has all infima and we know that a lattice has all infima if and only if it has all suprema. □

Lemma 9. *For $X \in$ **Locale**, there is an isomorphism of frames $\mathcal{O}(X) \cong Sub_{Sh(X)}(\mathbf{1})$.*

Proof. Since each representable functor is a sheaf for X and since $\mathcal{O}(X)(V, U) \in \{\emptyset, \{\star\}\}$, we have that $\mathcal{O}(X)(-, U)$ is a subsheaf of $\mathbf{1}$, so

$$y : \mathcal{O}(X) \to Sub_{Sh(X)}(\mathbf{1}) : U \mapsto \mathcal{O}(X)(-, U),$$

is well-defined. It is an injective frame morphism since

$$U \leq V \iff \mathcal{O}(X)(-, U) \subseteq \mathcal{O}(X)(-, V).$$

To show that it is surjective, let P be a subsheaf of $\mathbf{1}$. Let $U := \bigvee \{V \in \mathcal{O}(X) | P(V) = 1\}$. Since P is a sheaf, $P(U) = 1$, so by functoriality of P, if $V \leq U$, we have $P(V) = 1$. Thus $P(V) = 1$ if and only if $V \leq U$, which shows that $P = \mathcal{O}(X)(-, U)$. □

[1] See for example chapter 4, section 6 in [1]

4.2 Localic Giraud

In this subsection we characterize those grothendieck topoi which are localic using the theorem of Giraud. In order to show this, we need the following lemma:

Lemma 10. *Let $(\mathcal{C}, J)$ be a (small) site and denote by*

$$a : Func(\mathcal{C}^{op}, \mathbf{Set}) \to Sh_J(\mathcal{C}),$$

the sheafification functor and by

$$y : \mathcal{C} \to Func(\mathcal{C}^{op}, \mathbf{Set}),$$

the yoneda embedding. The set $\{ay(C)|C \in \mathcal{C}\}$ generates $Sh_J(\mathcal{C})$.

Proof. Let $F \in Sh_j(\mathcal{C})$ be a sheaf. Since each presheaf is a colimit of representable functors[2], we can write $F \cong colim_{C\in\mathcal{D}} y(C)$ (for some $\mathcal{D} \subseteq \mathcal{C}$). Thus

$$F \cong a(F) \cong a\left(colim_{C\in\mathcal{D}} y(C)\right) \cong colim_{C\in\mathcal{D}} ay(C),$$

where the last equality holds since a preserves colimits (as it is a left adjoint). Since each sheaf is such a colimit, the set $\{ay(C)|C \in \mathcal{C}\}$ indeed generates $Sh_J(\mathcal{C})$. □

Theorem 3. *("**Giraud's theorem for localic topoi**") Let $\mathcal{E}$ be a grothendieck topos. The following are equivalent:*

1. *$\mathcal{E}$ is localic.*
2. *There exists a site for $\mathcal{E}$ with a poset as underlying category.*
3. *$\mathcal{E}$ is generated by the subobjects of the terminal object $\mathbf{1}$.*

Proof. Since a frame is a poset, (1) $\implies$ (2) is clear. Assume (2), i.e. $\mathcal{E} = Sh(\mathbf{P}, J)$ for some poset category $\mathbf{P}$ and J a grothendieck topology on $\mathbf{P}$. Since $\mathbf{P}$ is a poset category, each hom-set is either empty or a singleton, thus all representable functors are subfunctors of the terminal presheaf $\mathbf{1}$, i.e. if we write

$$y : \mathbf{P} \to Fun(\mathbf{P}^{op}, \mathbf{Set}) : p \mapsto \mathbf{P}(-, p)$$

for the yoneda embedding, the unique morphism $y(p) \to \mathbf{1}$ (given by the componentwise inclusion) is a monomorphism for each $p \in \mathbf{P}$. Let

$$a : Fun(\mathbf{P}^{op}, \mathbf{Set}) \to Sh(\mathbf{P}, J)$$

be the sheafification functor. Since a preserves limits, it preserves monomorphisms[3] and the terminal object, so $ay(p) \to \mathbf{1}$ is also a monomorphism and thus $ay(p)$ is a subobject of $\mathbf{1}$. Using the previous lemma, we conclude (3).
Now assume (3). Recall that in the proof of the theorem of Giraud[4], the site that is constructed has as its underlying category the (small) category generated by the generators which by hypothesis is $Sub_{\mathcal{E}}(\mathbf{1})$. So (as $\mathcal{E}$ is a Grothendieck topos) $\mathcal{E} \cong Sh_J(Sub_{\mathcal{E}}(\mathbf{1}))$ for some grothendieck topology J. Since $Sub_{\mathcal{E}}(\mathbf{1})$ is a frame, it corresponds with a locale and thus shows the claim. □

[2]See for example Proposition I.5.1 in [1]

[3]a morphism $f : A \to B$ is mono if and only if its kernel pair is trivial, i.e. the pullback of $A \xrightarrow{f} B \xleftarrow{f} A$ is given by (A, Id_A, Id_A)

[4]Giraud's theorem gives a precise characterization of a category being a grothendieck topos in purely categorical properties.

4.3 From topoi to locales: Localic reflection

In this (sub)section we show that $X \mapsto Sh(X)$, for $X \in \textbf{Locale}$, induces a fully faithfull functor from **Locale** to **Topos**, which moreover has a left adjoint, called the **localic reflection**. These results are slighty more general in the sense that this actually happens at a 2-categorical level. So before we start proving these results, we first see how **Locale** and **Topos** become 2-categories:

- Let $f \in \textbf{Locale}(X, Y)$. Since $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ can be considered as a functor (as posets can be considered as categories), the set of locale morphisms $\textbf{Locale}(X, Y)$ can be given the structure of a category by defining the morphisms between locales maps to be the natural transformations between the corresponding frame morphisms.

- Let $f, g \in \textbf{Topos}(\mathcal{E}, \mathcal{F})$. So this means that we have functors $f^\star$ and $g^\star$. A 2-cell from f to g is then given by a natural transformation $f^\star \to g^\star$. Equivalently this is a natural transformation $g_\star \to f_\star$.

Remark 3. *To construct the* 2*-cells in* **Locale***, we have used its correspondance with* **Frm** *which automatically makes* **Frm** *into a* 2*-category and we have that* **Locale** *is the opposite* 2*-category of* **Frm** *in the sense that the* 1*-cells are reversed but the* 2*-cells are the same.*

Before continuing, we recall that given a site $(\mathcal{C}, J)$, a functor $\mathcal{C} \to \textbf{Set}$ is *continuous* if send covering sieves to epimorphic families.

Proposition 15. *Let $(\mathcal{C}, J)$ be a finitely complete site and $\mathcal{E}$ be (small) cocomplete topos. There is an equivalence of categories between geometric morphisms $\mathcal{E} \to Sh_J(\mathcal{C})$ and the category of continuous left exact functors $\mathcal{C} \to \mathcal{E}$ (with natural transformations as morphisms).*

Proof. This is corollary 4 of chapter 7 section 9 in [1]. □

Lemma 11. *Let $\mathcal{E}$ be a cocomplete elementary topos and Y a locale. The continuous left exact functors $\mathcal{O}(Y) \to \mathcal{E}$ are precisely the frame morphisms $\mathcal{O}(Y) \to Sub_{\mathcal{E}}(1)$.*

Proof. Consider a left exact continuous functor $F : \mathcal{O}(Y) \to \mathcal{E}$. Since Y is the terminal object in $\mathcal{O}(Y)$ and F is left exact, $F(Y) = \mathbf{1}_{\mathcal{E}}$ the terminal object in $\mathcal{E}$. And since $U \leq Y$ for each $U \in \mathcal{O}(Y)$, we have a morphism $F(U) \to F(Y) = \mathbf{1}_{\mathcal{E}}$. Since all morphisms in a poset category are monomorphisms and F is left exact, $F(U) \to \mathbf{1}_{\mathcal{E}}$ is a monomorphism. Thus we conclude that the image of F lies in $Sub_{\mathcal{E}}(\mathbf{1}_{\mathcal{E}})$.
Since $Sub_{\mathcal{E}}(\mathbf{1}_{\mathcal{E}})$ is also a complete Heyting algebra, left exactness of F means that F preserves finite infima and the continuity of F means that F preserves (arbitrary) suprema. So these continuous left exact functors are precisely the frame morphisms $\mathcal{O}(Y) \to Sub_{\mathcal{E}}(\mathbf{1}_{\mathcal{E}})$. □

By considering frames as (certain) functors, the frame morphisms form a category by considering natural transformations. The lemma combined with proposition (15) then becomes:

Corollary 6. *Let $\mathcal{E}$ be a cocomplete elementary topos and Y a locale. Then there is an equivalence of categories*

$$\textbf{ETopos}(\mathcal{E}, Sh(Y)) \to \textbf{Frm}(\mathcal{O}(Y), Sub_{\mathcal{E}}(\mathbf{1})).$$

Corollary 7. *Let X, Y be locales. Then there is an equivalence of categories*

$$\textbf{Locale}(X, Y) \to \textbf{Topos}(Sh(X), Sh(Y)).$$

Proof. Since $Sub_{Sh(X)}(\mathbf{1}) \cong \mathcal{O}(X)$, this follows from the previous corollary since there is an isomorphism of categories

$$\textbf{Locale}(X, Y) \cong \textbf{Frm}(\mathcal{O}(Y), \mathcal{O}(X)).$$

□

So in particular, this makes the assignment $X \mapsto Sh(X)$ into a fully faithfull functor.
Since $Sub_{\mathcal{E}}(1)$ is a frame, it corresponds with a locale $Loc(\mathcal{E})$, i.e. $\mathcal{O}(Loc(\mathcal{E})) = Sub_{\mathcal{E}}(1)$. So corollary (6) becomes:

Theorem 4. *("**Localic reflection of cocomplete elementary topoi**") Let $\mathcal{E}$ be a cocomplete topos and Y a locale. There is an equivalence of categories*

$$\mathbf{ETopos}(\mathcal{E}, Sh(Y)) \to \mathbf{Locale}(Loc(\mathcal{E}), Y).$$

So if we restrict to the Grothendieck topoi, which makes the assignment $\mathcal{E} \mapsto Loc(\mathcal{E})$ into a functor $Loc : \mathbf{Topos} \to \mathbf{Locale}$, the theorem implies that we have an adjunction $Loc \dashv Sh$.

4.4 Embeddings and surjections of localic topoi

In the previous section, we have seen that the morphisms between locales correspond with the geometric morphisms between their corresponding topoi. In this (sub)section, we show that this correspondance behaves well with respect to the embeddings and surjections.
Recall that a geometric morphism f is **surjective** if $f^\star$ is faithful and f is an **embedding** if $f_\star$ is fully faithfull.

Lemma 12. *Let $f : \mathcal{E} \to \mathcal{F}$ be an embedding. If $\{G_i\}_{i\in I}$ is a generating collection for $\mathcal{F}$, then is $\{f^\star(G_i)\}_{i\in I}$ a generating collection for $\mathcal{E}$.*

Proof. Assume $\alpha \neq \beta \in \mathcal{E}(E_1, E_2)$. Since f is an embedding, $f_\star$ is faithfull, thus $f_\star(\alpha) \neq f_\star(\beta)$. Since the G_i form a generating set, there exists some $u : G_i \to f_\star(E_1)$ such that $f_\star(\alpha) \circ u \neq f_\star(\beta) \circ u$.
Since $f^\star \dashv f_\star$, we have for the transpose $\hat{u} : f^\star(G_i) \to E_1$ of u, that $\alpha \circ \hat{u} \neq \beta \circ \hat{u}$. So the $f^\star(G_i)$ form indeed a generating set for $\mathcal{E}$. □

Corollary 8. *Let $f : \mathcal{E} \to \mathcal{F}$ be an embedding. If $\mathcal{F}$ is localic, so is $\mathcal{E}$.*

Proof. By Giraud, $Sub_{\mathcal{F}}(1_{\mathcal{F}})$ forms a generating set of $\mathcal{F}$. So by the lemma, $\{f^\star(S) | S \in Sub_{\mathcal{F}}(1_{\mathcal{F}})\}$ is a generating set of $\mathcal{E}$. So by Giraud, it suffices to show that $f^\star(S)$ is a subobject of $1_{\mathcal{E}}$, this is indeed the case because $f^\star$ is exact. □

To show that the surjections and embeddings of locales correspond with the surjections and embeddings of their corresponding localic topoi, we need the following lemma

Lemma 13. *Consider $f \in \mathbf{Locale}(X, Y)$ and $\tilde{f} := Sh(f) \in \mathbf{Topos}(Sh(X), Sh(Y))$ the corresponding geometric morphism. Identifying $\mathcal{O}(X)$ with $Sub_{Sh(X)}(\mathbf{1})$, we have:*

$$\tilde{f}_\star|_{\mathcal{O}(X)} = f_\star, \quad \tilde{f}^\star|_{\mathcal{O}(Y)} = f^{-1}.$$

Proof. Since each $V \in \mathcal{O}(Y)$ is identified with $\mathcal{O}(Y)(-, V)$, we conclude $\tilde{f}_\star|_{\mathcal{O}(X)} = f_\star$ from:

$$\tilde{f}_\star(\mathcal{O}(X)(-, U)) = \mathcal{O}(X)(-, U) \circ f^{-1} = \mathcal{O}(X)(f^{-1}(-), U) = \mathcal{O}(Y)(-, f_\star(U)).$$

From $\tilde{f}_\star|_{\mathcal{O}(X)} = f_\star$, the second equality $\tilde{f}^\star|_{\mathcal{O}(Y)} = f^{-1}$, indeed: The left adjoint is unique up to natural isomorphism, so we have for each $V \in \mathcal{O}(Y)$ an isomorphism $\eta_V : \tilde{f}^\star|_{\mathcal{O}(Y)}(V) \to f^{-1}(V)$. But these isomorphisms live in $\mathcal{O}(X)$ (a thin category), thus $\tilde{f}^\star|_{\mathcal{O}(Y)}(V) = f^{-1}(V)$ which shows the claim. □

Proposition 16. *Let $f \in \mathbf{Locale}(X, Y)$ and $\tilde{f} := Sh(f) \in \mathbf{Topos}(Sh(X), Sh(Y))$ the corresponding geometric morphism. Then is f a surjection (resp. embedding) of locales if and only if $\tilde{f}$ is a surjection (resp. an embedding) of topoi.*

Proof. We first show $\Leftarrow$. First assume that $\tilde{f}$ is a surjection, then (by definition) is $\tilde{f}^\star$ faithfull. In particular we have that $\tilde{f}^\star$ reflects isomorphisms (since we have topoi are balanced). So $f^{-1} = \tilde{f}^\star|_{\mathcal{O}(Y)}$ also reflects isomorphisms. Now assume that $f^{-1}(U) = f^{-1}(V)$, so

$$f^{-1}(U \wedge V) = f^{-1}(U) \wedge f^{-1}(V) = f^{-1}(U).$$

Since $U \wedge V \leq U$, we thus have that $U \wedge V = U$ (since f^{-1} reflects isomorphisms), so by symmetry we conclude $U = U \wedge V = V$.

Now assume $\tilde{f}$ is an embedding, then (by definition) is $\tilde{f}^\star$ fully faithfull. Since $\tilde{f}^\star \dashv \tilde{f}_\star$ and fully faithfullness, we have that the counit of this adjunction is an isomorphism, so by restricting to $\mathcal{O}(X)$, we have isomorphisms $U \cong f^{-1}f_\star U$ for all $U \in \mathcal{O}(X)$ which shows that f^{-1} is surjective, i.e. f is an embedding.
We now show $\Rightarrow$: Every geometric morphism can be factored as a surjection followed by an embedding (see for example theorem 6, section 4, chapter 7 in [1]). So we can write $\tilde{f}$ as follows:

$$\begin{array}{ccc} Sh(X) & \xrightarrow{\tilde{f}} & Sh(Y) \\ & \searrow^{\tilde{g}} & \uparrow{\scriptstyle \tilde{u}} \\ & & \mathcal{E} \end{array}$$

Since $Sh(Y)$ is localic, so is $\mathcal{E}$ (by corollary 8), i.e. $\mathcal{E} = Sh(Z)$ for some locale Z. Since we have the equivalence of categories $\mathbf{Locale}(X,Y) \cong \mathbf{Topos}(Sh(X), Sh(Y))$, there exists some $u : Z \to Y$ and $g : X \to Z$ such that

$$f = u \circ g, \quad Sh(u) = \tilde{u}, \quad Sh(g) = \tilde{g}.$$

From $\Leftarrow$, we conclude that g is a surjection and u is an embedding of locales. From this it is a standard argument that shows the claim, more precisely:
Since $f = u \circ g$, we have that if f is a surjection (resp. an embedding), that u is a surjection (resp. g an embedding) and thus u (resp. g) is an isomorphism. So we have that $\tilde{u} = Sh(u)$ (resp. $\tilde{g} = Sh(g)$) is an isomorphism (since Sh is a functor) from which it follows that f is a surjection (resp. an embedding) which concludes the proof of $\Rightarrow$. □

Corollary 9. *Let $X \in$ **Locale**. The sublocales of X correspond to the subtopoi of $Sh(X)$.*

Proof. By corollary (8), a subtopoi of $Sh(X)$ is localic. So a subtopos is given by an embedding $\tilde{f} \in \mathbf{Topos}(Sh(Y), Sh(X))$ which, by corollary (7) corresponds with a unique $f \in \mathbf{Locale}(Y, X)$. From the previous proposition, f is also an embedding so each subtopoi comes from a unique sublocale. □

References

[1] S. Mac Lane, I. Moerdijk *Sheaves in Geometry and Logic* 1992.