Project Category theory:
An introduction to Topos theory

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Abstract

In this project we study a certain kind of categories called *topoi*. A topos is a generalisation of the category **Set**. The goal of this project is to show that a topos has a great amount of categorical properties, more specifically that it is cartesian closed and that it is finitely cocomplete. But in order to show the latter, we introduce monadic functors, as they create limits and show Beck's theorem, which give a criteria for when a functor is monadic, which allows us to prove that the power object functor is monadic and hence create limits from which finite cocompleteness will follow.

1 Introduction

A topos is a generalisation of the category of sets like an abelian category is a generalisation of the category of abelian groups or R-modules. The purpose of abelian categories is to do homological algebra, but in topoi, the purpose is to interpret (propositional and predice) logic and so in particular interpret set theory. In set theory, the fundamental notion is that of the *membership relation*. This relation (on a set X) is a subset of $X \times \mathbb{P}X$. So to generalize this notion, we need the notions of subsets and powersets in general categories (with sufficient limits), which will be called subobjects resp. powerobjects. In this section we will generalize these notions to the level of category theory. But before introducing powerobjects, we need the notion of a subobject classifier. This is a generalization of the set $\{0,1\}$, with the purpose of classifying subobjects. In **Set**, this set classifies subsets by the bijection $\mathbb{P}(X) \to Hom(X,\{0,1\})$ which send a subset to its characteristic function.
We will see that a topos, just like **Set**, has a lot of (categorical) structure. It will have all finite limits and colimits, but also exponentials (i.e. the functions between sets is again a set). A category in which such exponentials always exists is called cartesian closed and so at the end of this section we will also introduce this.

1.1 Subobjects

In category theory we are always interested in objects up to isomorphism. In **Set**, an isomorphism is a bijection, so we can consider subsets $A, B \subseteq C$ as the same if there exists a bijection between them. Since a subset is defined element-wise, we can not really define the notion of a subset in a category, but a subset can always be considered as the image of an injection into C:
Let $\mathcal{C}$ be a category and $C \in \mathcal{C}$ an object. If $k \in Hom_{\mathcal{C}}(A, C)$ and $h \in Hom_{\mathcal{C}}(B, C)$ are monomorphisms, we say that k and h are equivalent if there is an isomorphism $\phi : A \to C$ in $\mathcal{C}$ such that $h \circ \phi = k$, i.e. the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "k"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

If k is equivalent to h, we denote $k \sim h$.
It is clear that $\sim$ forms an equivalence relation by definition of an isomorphism (since it has an inverse) and that the composition of isomorphisms is again an isomorphism.

Definition 1. *A* ***subobject*** *of an object $C \in \mathcal{C}$ is an equivalence class of monomorphisms ending in C. The collection of subobjects of C is denoted by $Sub_{\mathcal{C}}(C)$.*

Proposition 1. *Let $\mathcal{C}$ be a category, then $Sub_{\mathcal{C}}(C)$ has a partial ordering given as follows: Let $[f], [g] \in Sub_{\mathcal{C}}(C)$ be subobjects with representatives $f : A \to C$ and $g : B \to C$. Define $[f] \leq [g]$ if and only if there exists a morphism $h \in Hom_{\mathcal{C}}(A, B)$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

Proof. It is again clear that this indeed defines a partial ordering, we only check that it is well-defined, i.e. the order is independant of the choice of representative. Assume $f : A \to C \sim g : B \to C$ (by $\phi : A \to B$ an isomorphism in $\mathcal{C}$). If $[f] \leq [h]$ (with $h : D \to C$), then there exists a morphism $\psi : A \to D$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[dr, "f"] & D \arrow[d, "h"] \\ & C \end{tikzcd}$$

Since ϕ is invertible, we have $f \circ \phi^{-1} = g$, so the following diagram is commutative:

$$\begin{tikzcd} B \arrow[r, "\phi^{-1}"] \arrow[drr, "g"] & A \arrow[r, "\psi"] \arrow[dr, "f"] & D \arrow[d, "h"] \\ & & C \end{tikzcd}$$

This shows that $[g] \leq [h]$, so it is indeed well-defined. $\square$

Denote by **Pos** the category of poset categories with morphisms the functors between them. So by the previous proposition we have that $Sub_{\mathcal{C}}$ is an object in **Pos**.

Proposition 2. *Let $\mathcal{C}$ be a category and $f \in Hom_{\mathcal{C}}(A, C)$ a morphism. This induces a morphism in* **Pos** *(i.e. a functor)*

$$Sub_{\mathcal{C}}(f) : Sub_{\mathcal{C}}(C) \to Sub_{\mathcal{C}}(A).$$

Proof. Let $f \in Hom_{\mathcal{C}}(A, C)$ a morphism and $m : S \to C$ a monomorphism. Consider the following pullback square:

$$\begin{tikzcd} f^{\star}(S) \arrow[r] \arrow[d, "f^{\star}(m)"'] & S \arrow[d, "m"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

Notice that $f^{\star}(m)$ is a monomorphism (since m is mono and it is the pullback). We now claim that $Sub(f) : Sub(C) \to Sub(A) : m \mapsto f^{\star}(m)$ is a functor. Let $[h], [k] \in Sub_{\mathcal{C}}(C)$ with representatives $h : B_1 \to C, k : B_2 \to C$. We have to show that if $[h] \leq [k]$, then $[f^{\star}(h)] \leq [f^{\star}(k)]$. By definition we have the following pullback squares:

$$\begin{tikzcd} f^{\star}(B_1) \arrow[r] \arrow[d, "f^{\star}(h)"'] & B_1 \arrow[d, "h"] \\ A \arrow[r, "f"] & C \end{tikzcd} \qquad \begin{tikzcd} f^{\star}(B_2) \arrow[r] \arrow[d, "f^{\star}(k)"'] & B_2 \arrow[d, "k"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

Since $[h] \leq [k]$, there exists a morphism $\phi : B_1 \to B_2$ such that the following diagram is commutative:

$$\begin{tikzcd} B_1 \arrow[r, "\phi"] \arrow[dr, "h"] & B_2 \arrow[d, "k"] \\ & C \end{tikzcd}$$

Denote by ψ the morphism $f^\star(B_1) \to B_1$, then

$$k \circ \phi \circ \psi = h \circ \psi = f \circ f^\star(h).$$

So the following diagram commutes:

$$\begin{tikzcd}
f^\star(B_1) \arrow[drr, "\phi\circ\psi"] \arrow[ddr, "f^\star(h)"'] & & \\
& f^\star(B_2) \arrow[r] \arrow[d, "f^\star(k)"] & B_2 \arrow[d, "k"] \\
& A \arrow[r, "f"] & C
\end{tikzcd}$$

So by the universal property of the pullback (which defines $f^\star(k)$), there exists a unique morphism $f^\star(B_1) \xrightarrow{\mu} f^\star(B_2)$ which completes the diagram, so in particular we have the following commuting diagram:

$$\begin{tikzcd}
f^\star(B_1) \arrow[r, "\mu"] \arrow[dr, "f^\star(h)"'] & f^\star(B_2) \arrow[d, "f^\star(k)"] \\
& A
\end{tikzcd}$$

This shows that $[f^\star(h)] \leq [f^\star(k)]$ holds, so we are done. $\square$

Corollary 1. *Let $\mathcal{C}$ be a category and $f \in Hom_{\mathcal{C}}(A, B)$. The assignments*

$$\begin{array}{rcl} C & \mapsto & Sub_{\mathcal{C}}(C) \\ f & \mapsto & Sub_{\mathcal{C}}(f) \end{array}$$

*induces a functor $Sub_{\mathcal{C}} : \mathcal{C}^{op} \to$ **Pos**.*

Proof. We have to show that for each monomorphism $m : S \to C$:

$$\begin{array}{rcl} (Id_C)^\star(m) & = & m \\ (g \circ f)^\star(m) & = & f^\star(g^\star(m)) \quad \forall f \in Hom(A, B), g \in Hom(B, C) \end{array}$$

That the identity is preserved, consider the following pullback diagram:

$$\begin{tikzcd}
(Id_C)^\star(S) \arrow[r] \arrow[d, "(Id_C)^\star(m)"'] & S \arrow[d, "m"] \\
C \arrow[r, "Id_C"] & C
\end{tikzcd}$$

But the following diagram is also a pullback square:

$$\begin{tikzcd}
S \arrow[r, "Id_C"] \arrow[d, "m"] & S \arrow[d, "m"] \\
C \arrow[r, "Id_C"] & C
\end{tikzcd}$$

So by the universal property of the pullback (the pullback is unique up to isomorphism) there is a (unique) isomorphism $\phi : (Id_C)^\star(S) \to S$ such that the following diagram commutes:

$$\begin{tikzcd}
(Id_C)^\star(S) \arrow[r, "\phi"] \arrow[dr, "(Id_C)^\star(m)"'] & S \arrow[d, "m"] \\
& C
\end{tikzcd}$$

So $[m] = [(Id_C)^\star(m)]$.
We now show that the composition is preserved. Let $f \in Hom(A, B), g \in Hom(B, C)$. So by definition of $f^\star$ and $g^\star$ we have that the following diagram consists of pullbacks squares:

$$\begin{tikzcd}
f^\star(g^\star(S)) \arrow[r] \arrow[d, "f^\star(g^\star(m))"'] & g^\star(S) \arrow[r] \arrow[d, "g^\star(m)"] & S \arrow[d, "m"] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

Hence the full square is a pullback square. But by definition of $(g \circ f)^\star$, the following diagram is a pullback square:

$$\begin{tikzcd}
(g \circ f)^\star(S) \arrow[r] \arrow[d, "(g\circ f)^\star(m)"'] & S \arrow[d, "m"] \\
A \arrow[r, "g\circ f"] & C
\end{tikzcd}$$

So by the uniqueness of the pullback, there exists an isomorphism

$$(g \circ f)^\star(S) \xrightarrow{\phi} f^\star(g^\star(S))$$

such that the following diagram is commutative:

$$\begin{tikzcd}
(g \circ f)^\star(S) \arrow[r, "\phi"] \arrow[dr, "(g\circ f)^\star(m)"'] & f^\star(g^\star(S)) \arrow[d, "f^\star(g^\star(m))"] \\
& A
\end{tikzcd}$$

Thus $[f^\star(g^\star(m))] = [(g \circ f)^\star(m)]$. □

Notice that in general one does not have that $Sub_{\mathcal{C}}(C)$ is a set, therefore we needed that the functor $Sub_{\mathcal{C}}$ lands in **Pos**, but if it is indeed a set for all $C \in \mathcal{C}$, we call $\mathcal{C}$ *well-powered*:

Definition 2. *A category $\mathcal{C}$ is **well-powered** if for every object $C \in \mathcal{C}$, $Sub_{\mathcal{C}}(C)$ is a set.*

Corollary 2. *If $\mathcal{C}$ is well-powered, then $Sub_{\mathcal{C}}$ induces a functor*

$$Sub_{\mathcal{C}} : \mathcal{C}^{op} \to \mathbf{Set}.$$

1.2 Subobject classifier

A subset $S \subseteq B$ is uniquely determined by its characteristic morphism:

$$\phi : B \to \{0, 1\} : x \mapsto \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases}$$

So a 2-element set can classify subsets. A *subobject classifier* in a category is an object which also classify subobjects:

Definition 3. *Let $\mathcal{C}$ be a category with pullbacks and terminal object 1. A **subobject classifier** is an object $\Omega \in \mathcal{C}$ together with a morphism $true : 1 \to \Omega$ such that for each monomorphism $m \in Hom_{\mathcal{C}}(S, B)$, there is a unique morphism $\phi \in Hom_{\mathcal{C}}(B, \Omega)$ such that the following diagram is a pullback square:*

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "m"] & 1 \arrow[d, "true"] \\
B \arrow[r, "\phi"] & \Omega
\end{tikzcd}$$

We call ϕ the characteristic morphism of m.

Example 1. *In $\mathcal{C} = \mathbf{Set}$, the subobject classifier Ω is a 2-element set with $true : 1 \to \Omega$ any function (1 is a 1-element set in* **Set***, so true is necessairy constant). Usually one chooses $\Omega = \{0, 1\} = \{\perp, \top\}$ and $true \equiv 1 \equiv \top$ since these represent the truth-values.*

Proof. Define

$$\Omega := \{0, 1\}, \quad true : 1 \to \Omega : \star \mapsto 1.$$

Let $f : A \to B$ be an injective function. Define

$$\phi : B \to \{0, 1\} : b \mapsto \begin{cases} 1, & \text{if } f^{-1}(b) \neq \emptyset \\ 0, & \text{else} \end{cases}$$

Since for each $a \in A$, $a \in f^{-1}(f(a))$, we have $\phi \circ f(a) = 1$. This shows that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & 1 \arrow[d, "true"] \\ B \arrow[r, "\phi"] & \{0,1\} \end{tikzcd}$$

We now show that this diagram is a pullback square. Let X be a set, $g : X \to B$ be a function such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[drr, bend left] \arrow[dr, bend right, "g"'] & & \\ & A \arrow[r] \arrow[d, "f"] & 1 \arrow[d, "true"] \\ & B \arrow[r, "\phi"] & \{0,1\} \end{tikzcd}$$

So for each $x \in X$, we have $\phi \circ g(x) = 1$. So by definition of ϕ we have $f^{-1}(g(x)) \neq \emptyset$. So there exists $y_x \in f^{-1}(g(x))$, but f is injective, so this y_x is unique and we have $f(y_x) = g(x)$. So define

$$\psi : X \to A : x \mapsto y_x.$$

This function satisfies $f(\psi(x)) = f(y_x) = g(x)$. So g factors indeed through f. We now show that ψ is unique. Assume that $\kappa : X \to A$ satisfies $g = f \circ \kappa$. So

$$f \circ \psi = g = f \circ \kappa.$$

But f is injective, i.e. a monomorphism. So $\psi = \kappa$. $\square$

Proposition 3. *The subobject classifier of a category $\mathcal{C}$ (with pullbacks and terminal object) is unique up to isomorphism.*

Proof. Let $1 \xrightarrow{true_1} \Omega_1$ and $\xrightarrow{true_2} \Omega_2$ be subobject classifiers. Since 1 is terminal, every outgoing morphism is mono, so 1 is a subobject of every object, in particular for Ω_1 and Ω_2, so by definition of a subobject classifier there exists characteristic morphisms ϕ, ψ such that the following diagrams are pullbacks squares:

$$\begin{tikzcd} 1 \arrow[r, "Id_1"] \arrow[d, "true_1"] & 1 \arrow[d, "true_2"] \\ \Omega_1 \arrow[r, "\phi"] & \Omega_2 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "Id_1"] \arrow[d, "true_2"] & 1 \arrow[d, "true_1"] \\ \Omega_2 \arrow[r, "\psi"] & \Omega_1 \end{tikzcd}$$

So the following square is a pullback:

$$\begin{tikzcd} 1 \arrow[r, "Id_1"] \arrow[d, "true_1"] & 1 \arrow[d, "true_1"] \\ \Omega_1 \arrow[r, "\phi\circ\psi"] & \Omega_1 \end{tikzcd}$$

But the following diagram is also a pullback square:

$$\begin{tikzcd} 1 \arrow[r, "Id_1"] \arrow[d, "true_1"] & 1 \arrow[d, "true_1"] \\ \Omega_1 \arrow[r, "Id_{\Omega_1}"] & \Omega_1 \end{tikzcd}$$

But by definition of the subobject classifier, the characteristic morphism is unique, so $\phi \circ \psi = Id_{\Omega_1}$. In the same way we get $\psi \circ \phi = Id_{\Omega_2}$. Thus we get the desired isomorphism. $\square$

The following proposition shows that a subobject is uniquely defined by its characteristic morphism:

Proposition 4. *Let $\mathcal{C}$ be a category with all pullbacks, a terminal object and a subobject classifier. Monomorphisms $k \in Hom_{\mathcal{C}}(A, C)$ and $h \in Hom_{\mathcal{C}}(B, C)$ are equivalent if and only if they have the same characteristic morphism.*

Proof. If k and h are equivalent, there exists an isomorphism $\kappa : A \to B$ such that $k = h \circ \kappa$. Let $\phi : C \to \Omega$ be the characteristic morphism of k. So the following diagram is a pullback square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "k"] & 1 \arrow[d, "true"] \\ C \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Since $k = h \circ \kappa$, the following diagram is commutative:

$$\begin{tikzcd} B \arrow[r, "\kappa"] \arrow[dr, "h"] & A \arrow[r] \arrow[d, "k"] & 1 \arrow[d, "true"] \\ & C \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

We have to show that B is the pullback of $C \xrightarrow{\phi} \Omega \xleftarrow{true} 1$. Assume there exists morphisms $!_1 : Z \to 1$ and $f : Z \to C$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[drr, "!_Z"] \arrow[ddr, "f"'] & & \\ & B \arrow[r, "!_B"] \arrow[d, "h"] & 1 \arrow[d, "true"] \\ & C \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Since we already know that A is the pullback, we have a unique morphism $\alpha : Z \to A$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[drr, "!_Z"] \arrow[dr, "\alpha"] \arrow[ddr, "f"'] & & \\ & A \arrow[r, "!_A"] \arrow[d, "k"] & 1 \arrow[d, "true"] \\ & C \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Since $\kappa : B \to A$ is an isomorphism (in $\mathcal{C}$), $\kappa^{-1} \circ \alpha$ is a morphism from Z to B. And by all the previous commuting diagrams, we get that the following diagram is commutative:

So before we can conclude that B is indeed the pullback (and thus has characteristic morphism ϕ, we must show that $\kappa^{-1} \circ \alpha : Z \to B$ is the unique morphism which makes the diagram commute. But if $g : Z \to B$ is also a morphism which makes the diagram commute, we have that the following diagram commutes:

But from the uniqueness of the universal property of A (as the pullback), we must therefore have $\kappa \circ g = \alpha$, so $g = \kappa^{-1} \circ \kappa \circ g = \kappa^{-1} \circ \alpha$. So the morphism is indeed unique.

For the converse: Let k and h have the same characteristic morphism ϕ, i.e. the following diagrams are pullbacks squares:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "k"] & 1 \arrow[d, "true"] \\ C \arrow[r, "\phi"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d, "h"] & 1 \arrow[d, "true"] \\ C \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

But since the pullback is unique up to isomorphism, there exist an isomorphism $\kappa : A \to B$ such that $\kappa \circ h = k$. Since κ is an isomorphism, we therefore have that k and h are equivalent monomorphisms, so they represent the same subobject. □

Proposition 5. *If $\mathcal{C}$ is a locally small category with pullbacks, a terminal object and a subobject classifier. Then $\Omega \in \mathcal{C}$ is a subobject classifier if and only if there is an isomorphism*

$$\theta_C : Sub_{\mathcal{C}}(C) \to Hom_{\mathcal{C}}(C, \Omega),$$

natural in $C \in \mathcal{C}$. In particular, $Sub_{\mathcal{C}}$ is representable.

Proof. Assume $\mathcal{C}$ has a subobject classifier Ω. By the previous proposition, the assignment which maps a suboject $S \to X$ to its characteristic morphism $\phi_S : X \to \Omega$ is a bijection. So we have to show that for each morphism $f \in Hom_{\mathcal{C}}(D, C)$ the following diagram commutes:

$$\begin{tikzcd} Sub(C) \arrow[r, "\theta_C"] \arrow[d, "f^\star"] & Hom(C, \Omega) \arrow[d, "- \circ f"] \\ Sub(D) \arrow[r, "\theta_D"] & Hom(D, \Omega) \end{tikzcd}$$

i.e. for each monomorphism $m : A \to C \in Sub_{\mathcal{C}}(C)$, we need

$$\theta_D \circ f^\star(m) = \theta_C(m) \circ f.$$

We have that the following diagram is a pullbacksquare:

$$\begin{tikzcd} f^\star(A) \arrow[r] \arrow[d, "f^\star(m)"] & A \arrow[r] \arrow[d, "m"] & 1 \arrow[d, "true"] \\ D \arrow[r, "f"] & C \arrow[r, "\theta_C(m)"] & \Omega \end{tikzcd}$$

Indeed, the left square (resp. right) is the pullback by definition of $f^\star$ (resp. $\theta_C(m)$). But the characteristic morphism of $f^\star(m)$ is by definition $\theta_D(f^\star(m))$, i.e. the following pullback diagram is a pullback square:

$$\begin{tikzcd} f^\star(A) \arrow[r] \arrow[d, "f^\star(m)"] & 1 \arrow[d, "true"] \\ D \arrow[r, "\theta_D(f^\star(m))"'] & \Omega \end{tikzcd}$$

So the subobject $f^\star(m) : f^\star(A) \to D$ has characteristic morphisms $\theta_D(f^\star(m))$ and $\theta_C(m) \circ f$, so they are equal.
We now do the converse: Assume there exists an object $\Omega \in \mathcal{C}$ such that

$$\theta_C : Sub_{\mathcal{C}}(C) \to Hom_{\mathcal{C}}(C, \Omega),$$

is natural for all $C \in \mathcal{C}$ and an isomorphism. We claim that Ω is a subobject classifier. Let $m : S \to X$ be a subobject and let $\phi = \theta_X(m)$. We have to construct a morphism $true : 1 \to \Omega$ such that the following diagram commutes (for all m):

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"] & 1 \arrow[d, "true"] \\ X \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

The only morphism we know in $Hom(\Omega, \Omega)$ is Id_Ω, so define $t : \omega \to \Omega$ such that $\theta_\Omega(t) = Id_\Omega$ (which is possible since θ_Ω is a bijection). By the naturality of θ we have that the following diagram commutes:

$$\begin{tikzcd} Sub(\Omega) \arrow[r, "\theta_\Omega"] \arrow[d, "\phi^\star"] & Hom(\Omega, \Omega) \arrow[d, "-\circ\phi"] \\ Sub(X) \arrow[r, "\theta_X"] & Hom(X, \Omega) \end{tikzcd}$$

By applying this to $t \in Sub(\Omega)$, we get

$$\phi = Id \circ \phi = \theta_\Omega(t) \circ \phi = \theta_X \circ \phi^\star(t).$$

But by definition we have $\phi = \theta_X(m)$, so by the bijectiveness of θ_X, we have $\phi^\star(t) = m$. So by definition of $\phi^\star$, the following diagram is a pullback square:

$$\begin{tikzcd} \phi^\star(\omega) = S \arrow[r] \arrow[d, "m=\phi^\star(t)"] & \omega \arrow[d, "t"] \\ X \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

So if we can show that ω is the terminal object and that ϕ is the unique morphism which makes this diagram a pullback square, we have indeed that $t : \omega \to \Omega$ is the subobject classifier.
By taking $S = X$ and $m = Id_X$, we have a morphism $\alpha : X \to \omega$ such that the following diagram is a pullback square:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "Id"] & \omega \arrow[d, "t"] \\ X \arrow[r, "\theta_X(Id)"] & \Omega \end{tikzcd}$$

Let $\beta \in Hom(X, \omega)$ be a morphism, we have to show that $\beta = \alpha$.
By the naturality of θ and $t \circ \beta : X \to \omega$, we have that the following diagram commutes:

$$\begin{tikzcd} Sub(\Omega) \arrow[r, "\theta_\Omega"] \arrow[d, "(t\circ\beta)^\star"] & Hom(\Omega, \Omega) \arrow[d, "-\circ(t\circ\beta)"] \\ Sub(X) \arrow[r, "\theta_X"] & Hom(X, \Omega) \end{tikzcd}$$

By applying this to Id_Ω, we get $t\circ\beta = \theta_X \circ (t\circ\beta)^\star(Id)$. By the same reasoning we have $t\circ\alpha = \theta_X\circ(t\circ\alpha)^\star(Id)$. But since the pullback over the identity is again the identity, we have $(t \circ \alpha)^\star(Id) = Id = (t \circ \beta)^\star(Id)$. So

$$t \circ \beta = \theta_X \circ (t \circ \beta)^\star(Id) = \theta_X \circ Id = \theta_X \circ (t \circ \alpha)^\star(Id) = t \circ \alpha.$$

But t is a monomorphism, so $\alpha = \beta$. So ω is indeed the terminal object 1.
For the uniqueness of ϕ: Assume that the following diagrams are pullback squares:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "m"] & 1 \arrow[d, "t"] \\ X \arrow[r, "\phi"] & \Omega \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r] \arrow[d, "m"] & 1 \arrow[d, "t"] \\ X \arrow[r, "\psi"] & \Omega \end{tikzcd}$$

By definition we have defined that $\phi := \theta_X(m)$ and since $\psi \in Hom(X, \Omega)$, there exists a subbobject $n : T \to X \in Sub(X)$ such that $\psi = \theta_X(n)$. So we know (from the first part of the proof) that the following diagram is a pullback square:

$$\begin{tikzcd} T \arrow[r] \arrow[d, "n"] & 1 \arrow[d, "t"] \\ X \arrow[r, "\psi"] & \Omega \end{tikzcd}$$

So by uniqueness of the pullback, there exists an isomorphism $\kappa : S \to T$ such that the following diagram is commutative:

$$\begin{tikzcd} S \arrow[r, "\kappa"] \arrow[dr, "m"'] & T \arrow[d, "n"] \\ & X \end{tikzcd}$$

So by definition of a subobject we have that $m : S \to X$ and $n : T \to X$ represent the same subobject, so $\phi = \theta_X(m) = \theta_X(n) = \psi$. □

Corollary 3. *If $\mathcal{C}$ is a locally small with finite limits and a subobject classifier, then is $\mathcal{C}$ well-powered.*

Proof. Since $\mathcal{C}$ is locally small, $Hom_{\mathcal{C}}(C, \Omega)$ is a set. But there is a bijection $Sub_{\mathcal{C}}(C) \to Hom_{\mathcal{C}}(C, \Omega)$ for all $C \in \mathcal{C}$, so $Sub_{\mathcal{C}}(C)$ is in bijection with a set and is therefore a set itself. □

1.3 Powerobjects

In order to set theory in a topos, one also need the notion of a *powerobject*, a categorification of the notion of the powerset. Since a subset (of B) is usually defined in terms of a membership relation:

$$\in_B : B \times \mathbb{P}(B) \to \{0, 1\} : (b, A) \mapsto \begin{cases} 1, & \text{if } b \in A \\ 0, & \text{else} \end{cases}$$

Definition 4. *Let $\mathcal{C}$ be a category with pullbacks, a terminal object and a subobject classifier Ω. The **powerobject** of $B \in \mathcal{C}$ is an object $\mathbb{P}B$ together with a morphism $\in_B : B \times \mathbb{P}B \to \Omega$ such that for each morphism $f \in Hom_{\mathcal{C}}(B \times A, \Omega)$, there exists a unique morphism $g \in Hom_{\mathcal{C}}(A, \mathbb{P}B)$ such that the following diagram commutes:*

$$\begin{tikzcd} B \times A \arrow[r, "1 \times g"] \arrow[dr, "f"'] & B \times \mathbb{P}B \arrow[d, "\in_B"] \\ & \Omega \end{tikzcd}$$

Remark 1. *In the previous definition, the existence of products is used, but if a category has a terminal object and pullbacks, then $A \times B$ is the pullback of $A \to 1 \leftarrow B$.*

Proof. Denote by $A \times_1 B$ the pullback of $A \to 1 \leftarrow B$, so there exists morphisms $\pi_A : A \times_1 B \to A$ and $\pi_B : A \times_1 B \to B$ such that the following diagram is a pullback square:

$$\begin{tikzcd} A \times_1 B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"] & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd}$$

Let $Z \in \mathcal{C}$ be an object and $f \in Hom_{\mathcal{C}}(Z, A), g \in Hom_{\mathcal{C}}(Z, B)$ morphisms. Since 1 is terminal, the following diagram is commutative:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d] \\ A \arrow[r] & 1 \end{tikzcd}$$

Therefore, since $A \times_1 B$ is a pullback, so there exists a unique morphism $Z \to B \times_1 A$ such that the following diagram is commutative:

$$\begin{tikzcd}
& Z \arrow[d] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& A \times_1 B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"'] & \\
A & & B
\end{tikzcd}$$

So $A \times_1 B$ is the product of A and B in $\mathcal{C}$. $\square$

Example 2. *In $\mathcal{C} = \mathbf{Set}$, the powerobject of a set B is the powerset $\mathbb{P}(B)$ of B with $\in_B$ the membership relation of B.*

Proof. Let $f \in Hom_{\mathbf{Set}}(B \times A, \{0,1\})$. We have to show that there exists a unique functor $g : A \to \mathbb{P}B$ such that $f = \in_B \circ (Id_B \times g)$. Define

$$g : A \to \mathbb{P}B : a \mapsto \{b \in B | f(b,a) = 1\}.$$

So for each $(b,a) \in B \times A$:

$$\begin{aligned}
\in_B (Id_B \times g)(b,a) &= \in_B (b \times \{x \in B | f(x,a) = 1\}) \\
&= \begin{cases} 1, & f(b,a) = 1 \\ 0, & f(b,a) = 0 \end{cases} \\
&= f(b,a)
\end{aligned}$$

We now show that this g is unique. Assume $h : A \to \mathbb{P}B$ satisfies

$$\in_B \circ (Id_B \times h) = f.$$

So by definition of $\in_B$, we have (for each $b \in B, a \in A$):

$$f(b,a) = \begin{cases} 1, & \text{if } b \in g(a) \\ 0, & \text{else} \end{cases}$$

So $f(b,a) = 1 \iff b \in g(a)$. So

$$g(a) = \{b \in B | f(b,a) = 1\}.$$

$\square$

Proposition 6. *Let $\mathcal{C}$ be a category with pullbacks, a terminal object 1, powerobjects and a subobject classifier. The powerobject of 1 is the subobject classifier.*

Proof. Denote by p the natural isomorphism from $1 \times -$ to $Id_{\mathcal{C}}$. We will show this proposition by showing that the subobject classifier Ω together with $\in_1 = p_\Omega : 1 \times \Omega \to \Omega$ satisfies the universal property of $\mathbb{P}1$. Let $f \in Hom(1 \times A, \Omega)$, so we have to show that there exists a unique morphism $g \in Hom(A, \Omega)$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \times A \arrow[r, "(Id_1 \times g)"] \arrow[dr, "f"] & 1 \times \Omega \arrow[d, "p_\Omega"] \\
& \Omega
\end{tikzcd}$$

Notice that p_Ω is an isomorphism, so in particular a monomorphism. Let ϕ be its characteristic morphism, i.e. the following diagram is a pullback square:

$$\begin{tikzcd}
1 \times \Omega \arrow[r] \arrow[d, "p_\Omega"] & 1 \arrow[d, "true"] \\
\Omega \arrow[r, "\phi"] & \Omega
\end{tikzcd}$$

Denote by $!_X$ the unique morphism $X \to 1$ (for $X \in \mathcal{C}$). So from the commutativity (and that p_Ω is an isomorphism), we have $\phi = true \circ !_{1\times\Omega} \circ p_\Omega^{-1}$. So

$$\begin{aligned} \phi \circ f &= true \circ (!_{1\times\Omega} \circ p_\Omega^{-1}) \circ f \\ &= true \circ (!_\Omega \circ f), \quad \text{by uniqueness of } \Omega \to 1 \\ &= true \circ !_{1\times A}, \quad \text{by uniqueness of } 1 \times A \to 1 \end{aligned}$$

So the following diagram commutes:

$$\begin{tikzcd} 1 \times A \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\ & 1 \times \Omega \arrow[r] \arrow[d, "p_\Omega"] & 1 \arrow[d, "true"] \\ & \Omega \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

But the inner square is a pullback, so there exists a unique morphism $h \in Hom(1 \times A, 1 \times \Omega)$ which completes the diagram. We now claim that the following diagram commutes:

$$\begin{tikzcd} 1 \times A \arrow[rr, "(Id_1 \times p_\Omega \circ h \circ p_A^{-1})"] \arrow[dr, "f"] & & 1 \times \Omega \arrow[dl, "p_\Omega"] \\ & \Omega & \end{tikzcd}$$

Let $g := p_\Omega \circ h \circ p_A^{-1} \in Hom(A, \Omega)$. By the naturality of p, we have that the following diagram commutes:

$$\begin{tikzcd} 1 \times A \arrow[r, "p_A"] \arrow[d, "Id_1 \times g"] & A \arrow[d, "g"] \\ 1 \times \Omega \arrow[r, "p_\Omega"] & \Omega \end{tikzcd}$$

Thus

$$p_\Omega \circ (Id_1 \times g) = g \circ p_A = p_\Omega \circ h = f.$$

Thus the diagram indeed commutes. Assume $\tilde{g} \in Hom(A, \Omega)$ also satisfies $f = p_\Omega \circ (Id_1 \times \tilde{g})$. Since p_Ω is an isomorphism, we have

$$Id_1 \times \tilde{g} = p_\Omega^{-1} \circ f = Id_1 \times g.$$

So by projection, we have $g = \tilde{g}$. So Ω with p_Ω is indeed (isomorphic with) the powerobject of 1. $\square$

Proposition 7. *Let $\mathcal{C}$ be a locally small category with all pullbacks, a terminal object 1 and subobject classifier. An object $B \in \mathcal{C}$ has a powerobject $\mathbb{P}B$ if and only if there is a natural isomorphism*

$$Hom_{\mathcal{C}}(B \times -, \Omega) \cong Hom_{\mathcal{C}}(-, \mathbb{P}B),$$

i.e. $Hom_{\mathcal{C}}(B \times -, \Omega)$ is representable.

Proof. Assume $\mathcal{C}$ has powerobjects and let $B \in \mathcal{C}$. By definition of the powerobject we have to for each $f \in Hom_{\mathcal{C}}(B \times A, \Omega)$, there exists a unique morphism $g_f \in Hom_{\mathcal{C}}(A, \mathbb{P}B)$ such that $f = \in_B \circ (Id_B \times g_f)$. Define

$$\phi_A : Hom_{\mathcal{C}}(B \times A, \Omega) \to Hom_{\mathcal{C}}(A, \mathbb{P}B) : f \mapsto g_f.$$

Since g_f defines f uniquely, by $f = \in_B \circ (Id_B \times g_f)$, we have that ϕ_A is surjective and it is moreover injective because if $g_{f_1} = g_{f_2}$, then by the same formula we have $f_1 = f_2$. So ϕ_A is a bijection for all $A \in \mathcal{C}$. So it remains to show that for all $\alpha \in Hom_{\mathcal{C}}(A, C)$, the following diagram commutes:

$$\begin{tikzcd} Hom(B \times C, \Omega) \arrow[r, "\phi_C"] \arrow[d, "- \circ (Id_B \times \alpha)"] & Hom(C, \mathbb{P}B) \arrow[d, "- \circ \alpha"] \\ Hom(B \times A, \Omega) \arrow[r, "\phi_A"] & Hom(A, \mathbb{P}B) \end{tikzcd}$$

So we have to show that for each $f \in Hom(B \times C, \Omega)$ that

$$\phi_C(f) \circ \alpha = \phi_A(f \circ (Id_B \times \alpha)),$$

holds. Recall that $\phi_A(f \circ (Id_B \times \alpha))$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd} B \times A \arrow[r, "Id_B \times \phi_A(f \circ (Id_B \times \alpha))"] \arrow[dr, "f \circ (Id_B \times \alpha)"'] & B \times \mathbb{P}B \arrow[d, "\in_B"] \\ & \Omega \end{tikzcd}$$

Since $\phi_C(f)$ is the (unique) morphism such that $\in_B \circ (Id_B \times \phi_C(f)) = f$, we have:

$$f \circ (Id_B \times \alpha) = \in_B \circ (Id_B \times \phi_C(f)) \circ (Id_B \times \alpha) = \in_B \circ (Id_B \times \phi_C(f) \circ \alpha)$$

So if we replace $\phi_A(f \circ (Id_B \times \alpha))$ by $\phi_C(f) \circ \alpha$ in the diagram, it still commutes, because by uniqueness of $\phi_A(f \circ (Id_B \times \alpha))$, we have

$$\phi_A(f \circ (Id_B \times \alpha)) = \phi_C(f) \circ \alpha.$$

We now do the converse: Fix $B \in \mathcal{C}$, so there exists an object $\tilde{B} \in \mathcal{C}$ such that

$$\phi : Hom_{\mathcal{C}}(B \times -, \Omega) \to Hom_{\mathcal{C}}(-, \tilde{B}),$$

is a natural isomorphism. We now claim that $\tilde{B}$ is the powerobject of B. The only morphism in $Hom(\tilde{B}, \tilde{B})$ that we know is $Id_{\tilde{B}}$. Since $\phi_{\tilde{B}}$ is bijective, there exists a unique morphism $\in_B$: $B \times \tilde{B} \to \Omega$ such that $\phi_{\tilde{B}}(\in_B) = Id_{\tilde{B}}$. Let $f \in Hom_{\mathcal{C}}(B \times A, \Omega)$. We have to show that the following diagram commutes:

$$\begin{tikzcd} B \times A \arrow[rr, "Id_B \times \phi_A(f)"] \arrow[dr, "f"] & & B \times \tilde{B} \arrow[dl, "\in_B"] \\ & \Omega & \end{tikzcd}$$

Since ϕ_A is a bijection, it suffices to show that $\phi_A(f) = \phi_A(\in_B \circ (Id_B \times \phi_A(f)))$. This is indeed the case: By the naturality of ϕ, we have that the following diagram commutes:

$$\begin{tikzcd} Hom(B \times \tilde{B}, \Omega) \arrow[r, "\phi_{\tilde{B}}"] \arrow[d, "- \circ (Id_B \times \phi_A(f))"'] & Hom(\tilde{B}, \tilde{B}) \arrow[d, "- \circ \phi_A(f)"] \\ Hom(B \times A, \Omega) \arrow[r, "\phi_A"] & Hom(A, \tilde{B}) \end{tikzcd}$$

So by applying this to $\in_B$, we get:

$$\phi_A(\in_B \circ (Id_B \times \phi_A(f))) = \phi_{\tilde{B}}(\in_B) \circ \phi_A(f) = Id_{\tilde{B}} \circ \phi_A(f) = \phi_A(f),$$

which shows the claim. So we are left to show that $\phi_A(f)$ is the unique morphism which makes the following diagram commute:

$$\begin{tikzcd} B \times A \arrow[rr, "Id_B \times \phi_A(f)"] \arrow[dr, "f"'] & & B \times \tilde{B} \arrow[dl, "\in_B"] \\ & \Omega & \end{tikzcd}$$

Assume $\hat{g} : A \to \tilde{B}$ is another morphism which makes the diagram commute (so $\phi_A(f)$ is replaced by $\hat{g}$). Since ϕ_A is a bijection, we can write $\hat{g} = \phi_A(g)$ for some $g \in Hom(B \times A, \Omega)$. So

$$f = \in_B \circ (Id_B \times \phi_A(f)) = \in_B \circ ((Id_B \times \hat{g}) = \in_B \circ (Id_B \times \phi_A(g)) = g.$$

So $f = g$ and consequently $\phi_A(f) = \phi_A(g) = \hat{g}$. Thus, $\tilde{B}$ is indeed the powerobject of B. $\square$

Corollary 4. *Let $\mathcal{C}$ be a locally small category with pullbacks and a terminal object* 1. *Then has $\mathcal{C}$ a subobject classifier and powerobjects if and only there is a natural isomorphism*

$$Sub_{\mathcal{C}}(B \times -) \cong Hom_{\mathcal{C}}(-, \mathbb{P}B).$$

for every object $B \in \mathcal{C}$.

Proof. Let $\mathcal{C}$ have a subobject classifier Ω and powerobjects. Since $\mathcal{C}$ is locally small, we have natural isomorphisms

$$Sub_{\mathcal{C}}(-) \cong Hom_{\mathcal{C}}(-, \Omega)$$
$$Hom_{\mathcal{C}}(B \times -, \Omega) \cong Hom_{\mathcal{C}}(-, \mathbb{P}B)$$

So combining these we get what we wanted to show.
We now do the converse: Since $\mathbb{P}1$ is the subobject classifier, we have from the given natural isomorphism:

$$Sub(-) \cong Sub(1 \times -) \cong Hom(-, \mathbb{P}1) \cong Hom(-, \Omega)$$
$$Hom(A, \mathbb{P}B) = Sub(B \times A) \cong Sub((B \times A) \times 1) \cong Hom(B \times A, \mathbb{P}1) \cong Hom(B \times A, \Omega)$$

Thus Ω is the subobject classifier and $\mathbb{P}B$ is the powerobject of B (for all $B \in \mathcal{C}$) since $\mathcal{C}$ is locally small. □

1.4 Cartesian closed categories

A function $f : Y \to X$ is defined as some subset of $Y \times X$, so the collection of functions between 2 fixed sets form again a set. Thus it is not necessairy to include this as an axiom in set theory. We will see (in the section of *Exponentials in topoi*) that an *object consisting of the functions* always exists in a topos, so one gets it for free as in the case of **Set**. We call an object which *consists* of the functions an exponential object:

Definition 5. *Let $\mathcal{C}$ be a category which has all finite products. Let $X, Y \in \mathcal{C}$. The exponential of X with Y is an object Y^X together with an evaluation morphism $ev : Y^X \times X \to Y$ which is universal in the following sense: If Z is another object in $\mathcal{C}$ and $e : Z \times X \to Y$ is a morphism, then there exists a unique morphism $d : Z \to Y^X$ such that the following diagram commutes:*

$$\begin{tikzcd} Z \times X \arrow[r, "d \times Id_X"] \arrow[dr, "e"'] & Y^X \times X \arrow[d, "ev"] \\ & Y \end{tikzcd}$$

*We call $X \in \mathcal{C}$ **exponentiable** if for all $Y \in \mathcal{C}$, Y^X exists. A category $\mathcal{C}$ is **cartesian closed** if it has all finite products and every object is exponentiable.*

Example 3. *The category* **Set** *is cartesian closed.*

Proof. We will show that if a category has a subobject classifier and powerobjects, then it is cartesian closed from which the result follows, but we show it concretely:
Let X, Y be sets and let $Y^X := Hom_{\mathbf{Set}}(X, Y)$ and define

$$ev : Y^X \times X \to Y : (f, x) \mapsto f(x).$$

Consider a function $e : Z \times X \to Y$. Define

$$d : Z \to Y^X : z \mapsto e(z, \cdot).$$

So for each $z \in Z, x \in X$:

$$ev \circ (d \times Id_X)(z, x) = d(z)(x) = e(z, x),$$

which shows that e factors through ev. We now show that this factorization is unique: Assume that $c \in Hom(Z, Y^X)$ satisfies $e = ev \circ (c \times Id_X)$. So

$$d(z)(x) = e(z, x) = ev \circ (c \times Id_X)(z, x) = c(z)(x).$$

As this holds for all $x \in X$ and $z \in Z$, we have that $c = d$. □

Proposition 8. *Let $X \in \mathcal{C}$ be exponentiable. The assignment $Y \mapsto Y^X$ induces a (covariant) functor*

$$(-)^X : \mathcal{C} \to \mathcal{C}.$$

Proof. So we first have to define $f^X \in Hom_{\mathcal{C}}(Y^X, Z^X)$ for a morphism $f : Y \to Z$. Consider the following diagram:

$$\begin{tikzcd} Y^X \times X \arrow[r, "ev_Y"] & Y \arrow[d, "f"] \\ Z^X \times X \arrow[r, "ev_Z"] & Z \end{tikzcd}$$

So by the universal property of Z^X, there exists a unique morphism

$$f^X : Y^X \to Z^X$$

such that $(f^X \times Id_X)$ completes the diagram. We now show that this indeed defines a functor.
We first show that the identity is preserved, i.e. we have to show that for each $Y \in \mathcal{C}$, $(Id_Y)^X = Id_{Y^X}$. We clearly have that the following diagram commutes:

$$\begin{tikzcd} Y^X \times X \arrow[r, "ev_Y"] \arrow[d, "(Id_{Y^X} \times Id_X)"'] & Y \arrow[d, "Id_Y"] \\ Y^X \times X \arrow[r, "ev_Y"] & Y \end{tikzcd}$$

But by the universal property of Y^X, there exists a unique morphism $Y^X \to Y^X$ such that the previous diagram commutes and Id_{Y^X} satisfies this property, so by uniqueness we have $(Id_Y)^X = Id_{Y^X}$.
We now show that the composition is preserved, i.e. we have to show that for all morphisms $f \in Hom(Y, Z), g \in Hom(Z, W)$, we have $(g \circ f)^X = g^X \circ f^X$. By definition of $f^{(X)}$ and g^X, the following diagram commutes:

$$\begin{tikzcd} Y^X \times X \arrow[r, "ev_Y"] \arrow[d, "(f^X \times Id_X)"'] & Y \arrow[d, "f"] \\ Z^X \times X \arrow[r, "ev_Z"] & Z \end{tikzcd} \qquad \begin{tikzcd} Z^X \times X \arrow[r, "ev_Z"] \arrow[d, "(g^X \times Id_X)"'] & Z \arrow[d, "g"] \\ W^X \times X \arrow[r, "ev_W"] & W \end{tikzcd}$$

So the following diagram commutes:

$$\begin{tikzcd} Y^X \times X \arrow[r, "ev_Y"] \arrow[d, "(f^X \times Id_X)"'] & Y \arrow[d, "f"] \\ Z^X \times X \arrow[r, "ev_Z"] \arrow[d, "(g^X \times Id_X)"'] & Z \arrow[d, "g"] \\ W^X \times X \arrow[r, "ev_W"] & W \end{tikzcd}$$

So again by uniqueness we have $(g \circ f)^X = g^X \circ f^X$. $\square$

Recall that if $\mathcal{C}$ has binary products, we have (for $X \in \mathcal{C}$) that the assignment $Y \to Y \times X$ defines a (covariant) functor

$$- \times X : \mathcal{C} \to \mathcal{C},$$

where a morphism $f : Y \to Z$ is mapped to

$$f \times X = f \times Id_X : Y \times X \to Z \times X.$$

Proposition 9. *Let $\mathcal{C}$ be a category with binary products. An object $X \in \mathcal{C}$ is exponentiable if and only if the functor $- \times X : \mathcal{C} \to \mathcal{C}$ has a right adjoint.*

Proof. Let X be exponentiable, we claim that $(-)^X : \mathcal{C} \to \mathcal{C}$ is the right adjoint of $- \times X$. So we have to show that for all $Y, Z \in \mathcal{C}$ there exists a bijection

$$\phi_Y^Z : Hom(Y \times X, Z) \to Hom(Y, Z^X),$$

which is natural in X. We first construct ϕ_Y^Z. Let $f \in Hom(Y \times X, Z)$ be a morphism. By the universal property of the exponent Z^X (denote by ev to be the evaluation morphism), there exists a unique morphism $\tilde{f} : Y \times Z^X \to Z$ such that $f = ev \circ (\tilde{f} \times Id_X)$, i.e. the following diagram is commutative:

$$\begin{tikzcd}
Y \times X \arrow[r, "\tilde{f} \times Id_X"] \arrow[dr, "f"'] & Z^X \times X \arrow[d, "ev"] \\
& Z
\end{tikzcd}$$

So define $\phi_Y^Z(f) := \tilde{f}$. Since $\tilde{f}$ is unique, ϕ_Y^Z is well-defined. We now claim that ϕ_Y^Z is a bijection:

- Let $\phi_Y^Z(f) = \phi_Y^Z(g)$ for $f, g \in Hom(Y \times X, Z)$, so

  $$f = ev \circ (\phi_Y^Z(f) \times Id_X) = ev \circ (\phi_Y^Z(g) \times Id_X) = g.$$

  So ϕ_Y^Z is injective.

- Let $\tilde{f} \in Hom(Y \times Z^X, Z)$. Define $f := ev \circ (\tilde{f} \times Id_X)$. But $\phi_Y^Z(f)$ is the unique morphism such that $f = ev \circ (\phi_Y^Z(f) \times Id_X)$, so $\phi_Y^Z(f) = \tilde{f}$. So ϕ_Y^Z is surjective.

We now show that ϕ_Y^Z is natural in Y, so fix $Z \in \mathcal{C}$. Let $f \in Hom_{\mathcal{C}}(Y_1, Y_2)$, we have to show that the following diagram commutes:

$$\begin{tikzcd}
Hom(Y_2 \times X, Z) \arrow[r, "\phi_{Y_2}^Z"] \arrow[d, "-\circ(f \times Id)"'] & Hom(Y_2, Z^X) \arrow[d, "-\circ f"] \\
Hom(Y_1 \times X, Z) \arrow[r, "\phi_{Y_1}^Z"] & Hom(Y_1, Z^X)
\end{tikzcd}$$

Let $g \in Hom(Y_2 \times X, Z)$. By definition of $\phi_{Y_1}^Z$, we that $\phi_{Y_1}^Z(g \circ (f \times Id_X))$ is the unique morphism such that

$$ev \circ \phi_{Y_1}^Z(g \circ (f \times Id_X)) = g \circ (f \times Id_X),$$

i.e. the following diagram is commutative:

$$\begin{tikzcd}
Y_1 \times X \arrow[r, "\phi_{Y_1}^Z(g \circ (f \times Id_X)) \times Id_X"] \arrow[dr, "g \circ (f \times Id_X)"'] & Z^X \times X \arrow[d, "ev"] \\
& Z
\end{tikzcd}$$

And we know that $\phi_{Y_2}^Z(g)$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd}
Y_2 \times X \arrow[r, "\phi_{Y_2}^Z(g) \times Id_X"] \arrow[dr, "g"'] & Z^X \times X \arrow[d, "ev"] \\
& Z
\end{tikzcd}$$

So precomposing the latter diagram with $f \times Id_X$, we get the following (commuting) diagram:

$$\begin{tikzcd}
Y_1 \times X \arrow[r, "(f \circ \phi_{Y_2}^Z(g)) \times Id_X"] \arrow[dr, "g \circ (f \times Id_X)"'] & Z^X \times X \arrow[d, "ev"] \\
& Z
\end{tikzcd}$$

So by uniqueness of $\phi_{Y_1}^{Z}(g \circ (f \times Id_X)) \times Id_X$, we get

$$\phi_{Y_1}^{Z}(g \circ (f \times Id_X)) \times Id_X = (f \circ \phi_{Y_2}^{Z}(g)) \times Id_X.$$

So we indeed have that the naturality in Y.
We now show the naturality in Z. Let $f \in Hom_{\mathcal{C}}(Z_1, Z_2)$ and fix $Y \in \mathcal{C}$. So we have to show that the following diagram commutes:

$$\begin{tikzcd}
Hom(Y \times X, Z_1) \arrow[r, "\phi_Y^{Z_1}"] \arrow[d, "f\circ -"] & Hom(Y, Z_1^X) \arrow[d, "f^X \circ -"] \\
Hom(Y \times X, Z_2) \arrow[r, "\phi_Y^{Z_2}"] & Hom(Y, Z_2^X)
\end{tikzcd}$$

Let $g \in Hom(Y \times X, Z_1)$. So $\phi_Y^{Z_2}(f \circ g)$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd}
Y \times X \arrow[r, "\phi_Y^{Z_2}(f\circ g)\times Id_X"] \arrow[dr, "f\circ g"'] & Z_2^X \times X \arrow[d, "ev_2"] \\
 & Z_2
\end{tikzcd}$$

So we are going to use the same strategy as for the naturality in Y by showing that $f^X \circ \phi_Y^{Z_1}(g)$ also satisfies this commutativity. Write $ev_i : Z_i^X \times X \to Z_i$ for the evaluation morphism (for $i = 1, 2$).
We know that $\phi_Y^{Z_1}(g)$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd}
Y \times X \arrow[r, "\phi_Y^{Z_1}(g)\times Id_X"] \arrow[dr, "g"] & {Z_1}^X \times X \arrow[d, "ev_1"] \\
 & Z_1
\end{tikzcd}$$

We also know that f^X is (uniquely) defined such that the following diagram commutes:

$$\begin{tikzcd}
Z_1^X \times X \arrow[r, "ev_1"] \arrow[d, "f^X\times Id_X"'] & Z_1 \arrow[d, "f"] \\
Z_2^X \times X \arrow[r, "ev_2"] & Z_2
\end{tikzcd}$$

So combining this, we get the following commuting diagram:

$$\begin{tikzcd}
Y \times X \arrow[r, "\phi_Y^{Z_1}(g)\times Id_X"] \arrow[dr, "g"] & {Z_1}^X \times X \arrow[d, "ev_1"] \arrow[dr, "f^X\times Id_X"] & \\
 & Z_1 \arrow[d, "f"] & Z_2^X \times X \arrow[dl, "ev_2"] \\
 & Z_2 &
\end{tikzcd}$$

So from the commutativity of this diagram we get

$$f \circ g = ev_2 \circ ((f^X \circ \phi_Y^{Z_1})(g) \times Id_X).$$

This was what we needed to show the naturality in Z.
We now show the converse. Let $F : \mathcal{C} \to \mathcal{C}$ be the right adjoint of $- \times X : \mathcal{C} \times \mathcal{C}$. We are going to show that for each $Y \in \mathcal{C}$, $F(Y) = Y^X$ becomes the exponent. We first have to construct $ev \in Hom(F(Y) \times X, Y)$. So by the natural isomorphism, this should correspond to a morphism in $Hom(F(Y), F(Y))$, but the only morphism we know for sure in there is the identity, so take $ev := (\phi_{F(Y)}^{Y})^{-1}(Id_{F(Y)})$. So we have to show that for each morphism $e : Z \times X \to Y$, there exists a unique morphism $d : Z \to F(X)$ such that the following diagram is commutative:

$$\begin{tikzcd}
Z \times X \arrow[r, "d\times Id_X"] \arrow[dr, "e"'] & F(Y)\times X \arrow[d, "ev"] \\
& Y
\end{tikzcd}$$

Since $e \in Hom(Z \times X, Y)$, we have $\phi_Z^Y(e) \in Hom(Z, F(Y))$. We now claim that this plays the roll of d. By the bijectiveness of ϕ_Z^Y, it suffices to show that

$$\phi_Z^Y(e) = \phi_Z^Y(ev \circ (\phi_Z^Y(e) \times Id_X)).$$

By the naturality of ϕ_Z in Z, we have that the following diagram commutes:

$$\begin{tikzcd}
Hom(F(Y)\times X, Y)) \arrow[r, "\phi^Y_{F(Y)}"] \arrow[d, "-\circ(\phi_Z^Y(e)\times Id_X)"'] & Hom(F(Y), F(Y)) \arrow[d, "-\circ\phi_Z^Y(e)"] \\
Hom(Z\times X, Y) \arrow[r, "\phi_Z^Y"] & Hom(Z, F(Y))
\end{tikzcd}$$

Recall $ev = (\phi^Y_{F(Y)})^{-1}(Id_X)$, so if we apply the commutativity to ev, we get

$$\phi^Z_{F(Y)}(ev \circ (\phi_Z^Y(e) \times Id_X)) = \phi^Y_{F(Y)}(ev) \circ \phi_Z^Y(e) = \phi_Z^Y(e).$$

□

Proposition 10. *Let $\mathcal{C}$ be a cartesian closed category with pullbacks and a subobject classifier. For every object $B \in \mathcal{C}$, Ω^B is the powerobject for B.*

Proof. This is just by definition of the exponent Ω^B, indeed: Let $\in_B$: $B \times \Omega_B \to \Omega$ be the evaluation. Let $f \in Hom_{\mathcal{C}}(B \times A, \Omega)$. We have to show that there exists a unique morphism $g : A \to \Omega^B$ such that the following diagram commutes:

$$\begin{tikzcd}
B \times A \arrow[r, "Id_B\times g"] \arrow[dr, "f"'] & B\times\Omega^B \arrow[d, "\in_B"] \\
& \Omega
\end{tikzcd}$$

But $\in_B$ is the evaluation, so this map exists (uniquely) by the universal property of the exponent Ω_B. □

2 Elementary topoi

Definition 6. *An* ***(elementary) topos*** *is a category $\mathcal{E}$ with all pullbacks, a terminal object and a subobject classifier such that every object has a powerobject. The plural of topos is topoi.*

Example 4. *The category* **Set** *is a topos.*

Before we continue we introduce some notations:

Notation 1. *We have seen that there is a one-to-one correspondance between $Sub_{\mathcal{E}}(A)$ and $Hom(A, \Omega)$ where a subobject correspond to its characteristic morphism. If ϕ is the characteristic morphism of $m : S \to A \in Sub(A)$, we also write $char(m) := char(S) := \phi$. But by definition of the powerobject (and that $\mathbb{P}A \cong \Omega^A$), there is also a one-to-one correspondance between $Hom(A, \Omega)$ and $Hom(1, \mathbb{P}A)$. If $s \in Hom(1, \mathbb{P}A)$ corresponds to ϕ, we also write $s = \ulcorner\phi\urcorner$.*
By definition of the powerobject, there is a one-to-one correspondance between $Hom(B\times A, \Omega)$ and $Hom(A, \mathbb{P}B)$, we say that corresponding morphisms are $\mathbb{P}$-transpose of eachother.

Proposition 11. *A topos $\mathcal{E}$ is finitely complete.*

Proof. It is sufficient to show that $\mathcal{E}$ has finite products and equalizers. As proven before, binary products exists as it is the pullback over the terminal object and the empty product is the terminal, so we only have to show that equalizers exists. That they exists follows that $\mathcal{E}$ has products and pullbacks. Let $f, g \in Hom_{\mathcal{E}}(A, B)$. Define (E, e) as the pullback of $A \xrightarrow{f \times g} B \times B \xleftarrow{\Delta} B$, so the following diagram is a pullback square:

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "e"] & B \arrow[d, "\Delta"] \\ A \arrow[r, "(f,g)"] & B \times B \end{tikzcd}$$

We claim that (E, e) is the equalizer of f and g. That $fe = ge$ holds follow from the following calculation:

$$fe = \pi_1(f, g)e = \pi_1 \Delta d = d = \pi_2 \Delta d = \pi_2(f, g)e = ge.$$

Here we used the commutativity of the pullback square and the definition of the diagonal morphism.
So we are left to show that for all $h \in Hom(C, A)$ such that $fh = gh$, h factors through e. We have this factorisation (by universal property of the pullback) if there exists a morphism $\phi \in Hom(C, B)$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "h"] & B \arrow[d, "\Delta"] \\ A \arrow[r, "(f,g)"] & B \times B \end{tikzcd}$$

Since $f \circ h = g \circ h$, we have

$$\delta \circ f \circ h = (f \circ h, f \circ h) = (f \circ h, g \circ h),$$

so $\phi := f \circ h$ does indeed make the diagram commutes and thus there exists a unique morphism $\tilde{z} \in Hom(Z, E)$ such that $z = e \circ \tilde{z}$. $\square$

Corollary 5. *A category is a topos if and only if it has finite limits, a subobject classifier and powerobjects.*

Proposition 12. *The assignment $B \mapsto \mathbb{P}B$ induces a functor $\mathbb{P} : \mathcal{E}^{op} \to \mathcal{E}$.*

Proof. Let $h \in Hom(B, C)$. Since $\in_C \circ (h \times Id) \in Hom_{\mathcal{E}}(B \times \mathbb{P}C, \Omega)$, there exists a unique morphism $g_h : \mathbb{P}C \to \mathbb{P}B$ (by definition the power object) such that the following diagram commutes:

$$\begin{tikzcd} B \times \mathbb{P}C \arrow[r, "Id \times g_h"] \arrow[d, "h \times Id"] & B \times \mathbb{P}B \arrow[d, "\in_B"] \\ C \times \mathbb{P}C \arrow[r, "\in_C"] & \Omega \end{tikzcd}$$

Define $\mathbb{P}(h) := g_h$.
Let $B = C$ and $h = Id_B$, we now claim that $\mathbb{P}(Id_B) := g_{Id_B} = Id_{\mathbb{P}B}$. This is indeed the case because the following diagram commutes:

$$\begin{tikzcd} B \times \mathbb{P}B \arrow[r, "Id \times Id"] \arrow[d, "Id \times Id"] & B \times \mathbb{P}B \arrow[d, "\in_B"] \\ B \times \mathbb{P}B \arrow[r, "\in_B"] & \Omega \end{tikzcd}$$

and because g_{Id_B} is the unique morphism which makes it commutate. But we see that $Id_{\mathbb{P}B}$ makes it also commutative, so they are equal.
Let $f \in Hom(A, B)$ and $h \in Hom(B, C)$. We have to show that $\mathbb{P}(h \circ f) = \mathbb{P}(f) \circ \mathbb{P}(h)$. So by definition of $\mathbb{P}(h \circ f)$, we have to show:

$$\in_C \circ ((h \circ f) \times Id) = \in_A (Id \circ (\mathbb{P}f \mathbb{P}h)).$$

This follows from the following calculation:

$$\begin{aligned}
\in_C \circ((h\circ f)\times Id) &= \in_C \circ(h\times Id)\circ(f\times Id) \\
&= \in_B \circ(Id\times \mathbb{P}h)\circ(f\times Id), \qquad \text{definition } \mathbb{P}h \\
&= \in_B \circ(f\times \mathbb{P}h) \\
&= \in_B \circ(f\times Id)\circ(Id\times \mathbb{P}h) \\
&= \in_A \circ(Id\times \mathbb{P}f)\circ(Id\times \mathbb{P}h), \qquad \text{definition } \mathbb{P}f \\
&= \in_A \circ(Id\times(\mathbb{P}f\circ \mathbb{P}h))
\end{aligned}$$

□

Recall that the diagonal morphism $\Delta_B : B \to B\times B$ is a monomorphism:

Definition 7. *Let $B\in\mathcal{E}$ and $\delta_B := char(\Delta_B)$ be the characteristic morphism of the diagonal morphism. The **singleton morphism** of B is the $\mathbb{P}$-transpose of δ_B and is denoted by $\{\cdot\}_B \in Hom(B,\mathbb{P}B)$.*

Example 5. *If $\mathcal{E} = \mathbf{Set}$, then for any set B, the singleton morphism is given by:*

$$\{\cdot\}_B : B \to \mathbb{P}(B) : b \mapsto \{b\}.$$

Proof. The diagonal of a set B (in **Set**) is $\Delta_B : B\to B\times B : b\mapsto (b,b)$. From the proof that **Set** has a subobject classifier, we get that the characteristic morphism δ_B of Δ_B is given by

$$\delta_B : B\times B \to \{0,1\} : (b_1,b_2)\mapsto \begin{cases} 1, & \text{if } b_1 = b_2 \\ 0, & \text{else} \end{cases}$$

From the proof that **Set** has powerobjects, we get that the $\mathbb{P}$-transpose of δ_B is given by

$$\{\cdot\}_B : B\to \mathbb{P}(B) : b\mapsto \{\tilde{b}\in B | \tilde{b} = b\} = \{b\}.$$

□

Lemma 1. *Let $B\in\mathcal{E}$. The singleton morphism $\{\cdot\}_B$ is a monomorphism.*

Proof. Let $b_1, b_2 \in Hom(X,B)$ such that $\{\cdot\}_B\circ b_1 = \{\cdot\}_B\circ b_2$. So their $\mathbb{P}$-transposes are equal, i.e.

$$\delta_B(Id\times b_1) = \delta_B(Id\times b_2).$$

By definition of δ_B, the following diagram is a pullback square:

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\Delta_B"] & 1 \arrow[d, "true"] \\
B\times B \arrow[r, "\delta_B"] & \Omega
\end{tikzcd}$$

We now claim that (for $i = 1,2$)

$$\begin{tikzcd}
X \arrow[r, "b_i"] \arrow[d, "b_i\times Id_X"'] & B \arrow[d, "\Delta_B"] \\
B\times X \arrow[r, "Id_B\times b_i"] & B\times B
\end{tikzcd}$$

is a pullback square. It clearly commutes. So assume that there exists morphisms $f_1\in Hom(Y, B\times X)$ and $f_2 \in Hom(Y,B)$ (for some $B\in\mathcal{C}$) such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[drr, bend left, "f_2"] \arrow[dr, bend right, "f_1"'] & & \\
& X \arrow[r, "b_i"] \arrow[d, "b_i\times Id_X"] & B \arrow[d, "\Delta_B"] \\
& B\times X \arrow[r, "Id_B\times b_i"] & B\times B
\end{tikzcd}$$

We will now show that $g := pr_X \circ f_1 : Y \to X$ is the unique morphism which completes the diagram. The commutativity of the diagram means:

$$(f_2, f_2) = \Delta_B \circ f_2 = (Id_B \times b) \circ f_1 = (pr_B \circ f_1, b \circ pr_X \circ f_1),$$

where $pr_B : B \times X \to B$ and $pr_X : B \times X \to X$ are the projections. Let $pr_1 : B \times B \to B$ be the projection on the first B and $pr_2 : B \times B \to B$ be the projection on the second. So from the commutativity we get:

$$\begin{cases} f_2 = pr_1 \circ (Id_B \times b_i) \circ f_1 = pr_1 \circ f_1 \\ f_2 = pr_2 \circ (Id_B \times b_i) \circ f_1 = b_i \circ pr_X \circ f_1 \end{cases}$$

Notice that the first equation implies that

$$f_1 = (pr_B \circ f_1, pr_X \circ f_1) = (f_2, pr_X \circ f_1).$$

So

$$(b_i \times Id_X) \circ pr_X \circ f_1 = (b \circ pr_X \circ f_1, pr_X \circ f_1) = (f_2, pr_X \circ f_1) = f_1.$$

And by the commutativity we had $f_2 = b \circ pr_X \circ f_1$. So we have that $g := pr_X \circ f_1$ factorizes f_1 (resp. f_2) through $(b \times Id_X)$ (resp. b). We now claim that this g is unique. Assume there exists a morphism $h \in Hom(Y, X)$ which also completes the diagram, then we have

$$g = pr_X \circ f_1 = pr_X \circ (b \times Id_X) \circ h = h.$$

Thus we have show that

$$\begin{tikzcd} X \arrow[r, "b_i"] \arrow[d, "b_i \times Id_X"'] & B \arrow[d, "\Delta_B"] \\ B \times X \arrow[r, "Id_B \times b_i"] & B \times B \end{tikzcd}$$

is indeed a pullback square for both $i = 1$ and $i = 2$. So we have that

$$\begin{tikzcd} X \arrow[r, "b_i"] \arrow[d, "b_i \times Id_X"'] & B \arrow[r] \arrow[d, "\Delta_B"] & 1 \arrow[d, "true"] \\ B \times X \arrow[r, "Id_B \times b_i"] & B \times B \arrow[r, "\delta_B"] & \Omega \end{tikzcd}$$

is a pullback diagram. But

$$\delta_B(Id \times b_1) = \delta_B(Id \times b_2),$$

so $b_1 \times Id_X : X \to B \times X$ and $b_2 \times Id_X : X \to B \times X$ have the same characteristic morphism, thus there exists a (unique) morphism $h : X \to X$ such that

$$b_1 \times Id_X = (b_2 \times Id_X) \circ h = (b_2 \circ h) \times h.$$

So by projecting on the second component we have $h = Id_X$ and thus by projection on the first component we have $b_1 = b_2 \circ h = b_2$. Thus we indeed have that $\{\cdot\}_B$ is a monomorphism. $\square$

Lemma 2. *Let $\mathcal{C}$ be a category and $f \in Hom_{\mathcal{C}}(X, Y)$ be an equalizer which is an epimorphism, then it is an isomorphism (dually a coequalizer and monomorphism is an isomorphism).*

Proof. We only show the case that an equalizer which is a epimorphism is an isomorphism. Assume that the following diagram is an equalizer diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & Z \end{tikzcd}$$

So $g \circ f = h \circ f$. But f is an epimorphism, so $g = h$. But the equalizer of g and g is Id_Y. So there exists a unique isomorphism $k \in Hom(Y, X)$ such that $f \circ k = Id_Y$, thus $f = Id_Y \circ k^{-1}$ but both Id_Y and k^{-1} are isomorphisms, so f is an isomorphism. $\square$

Proposition 13. *A topos is balanced, i.e. every monomorphism which is an epimorphism is an isomorphism.*

Proof. We first show that any monomorphism $m : S \to X$ in a topos $\mathcal{E}$ is the equalizer of $char(m)$ (the characteristic morphism of m) and $true_X = true \circ !_X$. Assume that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "f"] & X \arrow[r, shift left, "char(m)"] \arrow[r, shift right, "true_X"'] & \Omega \end{tikzcd}$$

So (using that there exists only a unique morphism into the terminal object) we have that the following diagram commutes:

$$\begin{tikzcd} T \arrow[drr, bend left, "!_T"] \arrow[ddr, bend right, "f"'] & & \\ & S \arrow[r, "!_S"] \arrow[d, "m"] & 1 \arrow[d, "true"] \\ & X \arrow[r, "char(m)"] & \Omega \end{tikzcd}$$

As the square is a pullback square (by definition of $char(m)$), there exists a unique morphism $g : T \to S$ which completes the diagram, i.e. $f = m \circ q$ which shows that m is the equalizer.
So by the previous lemma, we have that $\mathcal{E}$ is balanced. □

3 Exponentials in topoi

In this section we show that every object in a topos $\mathcal{E}$ is exponentiable and consequently gives a characterisation of topoi.

Theorem 1. *Every topos $\mathcal{E}$ is cartesian closed.*

Proof. Let $C, B \in \mathcal{E}$. We first construct the object C^B as follows: Define

$$v : B \times \mathbb{P}(C \times B) \to \mathbb{P}C$$

as the $\mathbb{P}$-transpose of

$$\in_{B\times C}: (C \times B) \times \mathbb{P}(C \times B) \to \Omega.$$

Denote by $\sigma_C : \mathbb{P}C \to \Omega$ the characteristic morphism of the singleton arrow $\{\cdot\}_C : C \to \mathbb{P}C$ and let $u : \mathbb{P}(C \times B) \to \mathbb{P}B$ be the $\mathbb{P}$-transpose of $\sigma_C v$.
Define C^B such that the following diagram is a pullback square:

$$\begin{tikzcd} C^B \arrow[r] \arrow[d, "m"] & 1 \arrow[d, "true_B"] \\ \mathbb{P}(C \times B) \arrow[r, "u"] & \mathbb{P}B \end{tikzcd}$$

i.e. it is the pullback of u along $true_B$. We now define the evaluation morphism $e : B \times C^B \to C$: Consider the following commutative diagram:

$$\begin{tikzcd} B \times C^B \arrow[r, "Id\times m"] \arrow[d, "Id\times !"] & B \times \mathbb{P}(C \times B) \arrow[r, "v"] \arrow[d, "Id\times u"] & \mathbb{P}C \arrow[d, "\sigma_C"] & C \arrow[l, "\{\cdot\}_C"] \arrow[d] \\ B \times 1 \arrow[r, "Id\times\lceil true_B \rceil"] & B \times \mathbb{P}B \arrow[r, "\in_B"] & \Omega & 1 \arrow[l, "true"] \end{tikzcd}$$

The left square commutes by definition of C^B. Since u is the $\mathbb{P}$-transpose of $\sigma_C v$, we have $\in_B \circ(Id \times u) = \sigma_C v$, so the middle square commutes. Since σ_C is the characteristic morphism of $\{\cdot\}_C$, the right square is a pullback square. Since $\lceil true_B \rceil$ is the characteristic morphism of $true_B : B \to 1 \xrightarrow{true} \Omega$, we have $\in_B \circ(Id \times \lceil true_B \rceil) = true_B$, so the (unique) morphism $B \times 1 \to 1$ still makes the diagram commutative. So there are morphisms $B \times C^B \to \mathbb{P}C$ and $B \times C^B$ which by the universal property of the pullback makes sure that there exists a unique morphism $e : B \times C^B \to C$.
We now show that (C^B, e) is indeed the exponential. Assume $f : B \times C \to A$ is a morphism in $\mathcal{E}$, so we have to show that there exists a unique morphism $g : A \to C^B$ such that $f = e(1 \times g)$. Consider $f : B \times C \to A$ and let $h : A \to \mathbb{P}(C \times B)$ be its $\mathbb{P}$-transpose, so

$$\delta_C(Id \times f) = \in_{C\times B} (Id \times Id \times h) : C \times B \times A \to \Omega.$$

By $\mathbb{P}$-transposing (using that δ_C (resp. v) is the $\mathbb{P}$-transpose of δ_C (resp. $\in_C \times B$), this equation becomes

$$\{\cdot\}_C \circ f = v(Id \times h) : B \times A \to \mathbb{P}C.$$

Since σ_C is the characteristic morphism of $\{\}_C$, $true_C = \sigma_C \circ \{\cdot\}_C$, so

$$true_C \circ f = \sigma_C \circ \{\cdot\}_C \circ f = \sigma_C \circ v \circ (Id \times h) : B \times A \to \Omega.$$

The composition of a morphism with $true$, is again the morphism $true$, so $true_C \circ f = true_{B\times A} = true_B \circ pr_B$ (where $pr_B : B \times A \to B$ is the projection). By taking the $\mathbb{P}$-transpose of $true_B \circ pr_B = \sigma_C \circ v \circ (Id \times h)$ (and again using that u is the $\mathbb{P}$-transpose of $\sigma_C \circ v$) we get

$$\lceil true_B \rceil \circ !_A = uh.$$

So by this equality (and the definition of C^B) the following diagram is commutative:

$$\begin{tikzcd}
A \arrow[drr, bend left, "h"] \arrow[ddr, bend right] & & \\
& C^B \arrow[r, "m"] \arrow[d] & \mathbb{P}(C \times B) \arrow[d, "u"] \\
& 1 \arrow[r, "\lceil true_B \rceil"] & \mathbb{P}B
\end{tikzcd}$$

But the square is a pullback square, so h factors through m by some morphism $g : A \to C^B$, i.e. $h = mg$. We now claim that $f = e \circ (Id \times g)$ holds. Because $\{\cdot\}_C$ is a monomorphism, this follows from the following calculation:

$$\{\cdot\}_C \circ f = v \circ (Id \times h) = v \circ (Id \times m) \circ (Id \times g) = \{\cdot\}_C \circ e \circ (Id \times g).$$

It remains now to prove the uniqueness. By the existence of g we have

$$\delta_C(Id \times f) = \in_{C\times B} (Id \times Id \times h) = \in_{C\times B} (Id \times Id \times mg).$$

So mg defines f uniquely. But if $\tilde{g} : A \to C^B$ would also satisfy this condition, we have $mg = m\tilde{g}$, but m is a monomorphism, so $g = \tilde{g}$, so g is the unique morphism. □

The previous theorems now give a characterisation of topoi:

Corollary 6. *A category $\mathcal{E}$ is a topos if and only if it is cartesian closed with all equalizers and has a subobject classifier.*

Proof. If $\mathcal{E}$ is a topos we have that it is cartesian closed by the previous theorem and it has all equalizers because we have shown that it is finitely cocomplete. For the converse: Since it has products and equalizers is it finitely complete, so we only have to show that it has powerobjects. In the preliminaries we have shown that if a category is cartesian closed, finitely complete and has a subobject classifier, then is Ω^B the powerobject for all $B \in \mathcal{C}$. □

4 Direct image

In this section we define the direct image of a monomorphism $k : B^{'} \to B$ in a topos $\mathcal{E}$.
Define $U_{B'} \in \mathcal{E}$ and $u_{B'} \in Hom_{\mathcal{E}}(U_{B'}, B^{'} \times \mathbb{P}B^{'})$ such that the following diagram is a pullback square:

$$\begin{tikzcd}
U_{B'} \arrow[r] \arrow[d, "u_{B'}"] & 1 \arrow[d, "true"] \\
B^{'} \times \mathbb{P}B^{'} \arrow[r, "\in_{B'}"] & \Omega
\end{tikzcd}$$

As this is a pullback and $true : 1 \to \Omega$ is a mono, $u_{B'}$ is a mono, so $U_{B'}$ is indeed a subobject of $B^{'} \times \mathbb{P}B^{'}$. Since k is a mono, $B^{'} \times \mathbb{P}B^{'}$ is a subobject of $B \times \mathbb{P}B^{'}$ (by $k \times Id$), so consequently $(k \times Id) \circ u_{B'}$ makes $U_{B'}$ into a subobject of $B \times \mathbb{P}$. Define $e_k : B \times \mathbb{P}B^{'} \to$ to be the characteristic morphism of $(k \times Id) \circ u_{B'}$, i.e. the following diagram is a pullback square:

$$\begin{tikzcd}
U_{B'} \arrow[r] \arrow[d, "u_{B'}"] & 1 \arrow[r, "Id"] \arrow[d, "true"] & 1 \arrow[dd, "true"] \\
B^{'} \times \mathbb{P}B^{'} \arrow[r, "\in_{B'}"] \arrow[d, "k \times Id"] & \Omega & \\
B \times \mathbb{P}B^{'} \arrow[rr, "e_k"] & & \Omega
\end{tikzcd}$$

Definition 8. *Let $k \in Hom_{\mathcal{E}}(B^{'}, B)$ be a monomorphism. Define*

$$\mathbb{P}B^{'} \xrightarrow{\exists_k} \mathbb{P}B$$

to be the $\mathbb{P}$-tranpose of e_k.

Example 6. *In $\mathcal{E} := \mathbf{Set}$, we have for every injection $k : B^{'} \to B$:*

$$\exists_k : \mathbb{P}(B^{'}) \to \mathbb{P}(B) : S^{'} \mapsto k(S^{'}) = \left\{ b \in B | \exists b^{'} \in S^{'} : k(b^{'}) = b \right\}.$$

Proof. Since

$$\in_{B'}: B^{'} \times \mathbb{P}(B^{'}) \to \{0,1\} : (b^{'}, S^{'}) \mapsto \begin{cases} 1, & \text{if } b^{'} \in S^{'} \\ 0, & \text{else} \end{cases}$$

the pullback of

$$\begin{tikzcd}
 & 1 \arrow[d, "true"] \\
B^{'} \times \mathbb{P}(B^{'}) \arrow[r, "\in_{B'}"] & \{0,1\}
\end{tikzcd}$$

is given by the subset

$$U_{B'} := \left\{ (b^{'}, S^{'}) \in B^{'} \times \mathbb{P}(B^{'}) | b^{'} \in S^{'} \right\}$$

with $u_{B'} : U_{B'} \to B^{'} \times \mathbb{P}(B^{'})$ the inclusion function. So the characteristic morphism of

$$(k \times Id) \circ u_{B'} : U_{B'} \to B^{'} \times \mathbb{P}(B^{'}) \to B \times \mathbb{P}(B^{'}) : (b^{'}, S^{'}) \mapsto (k(b^{'}), S^{'}),$$

is given by

$$e_k : B \times \mathbb{P}(B^{'}) \to \{0,1\} : (b, S^{'}) \mapsto \begin{cases} 1, & \text{if } \exists b^{'} \in S^{'} \text{ s.t. } b = k(b^{'}) \\ 0, & \text{else} \end{cases}$$

Thus the $\mathbb{P}$-transpose of e_k is indeed given by what we wanted to show. $\square$

Proposition 14. *Let $S \xrightarrow{m} B' \xrightarrow{k} B$ be monics in $\mathcal{E}$. Then*

$$\exists_k \lceil char(m) \rceil = \lceil char(km) \rceil : 1 \to \mathbb{P}B.$$

Proof. By $\mathbb{P}$-transposing, we have to show

$$e_k(Id \times \lceil char(m) \rceil) = char(km) : B \cong B \times 1 \to \Omega.$$

Since these morphisms are predicates, they characterize a subobject of B, so to show that this equality holds, we have to show that they characterize the same subobject.
We have (by definition of k and m) that S is the subobject with characteristic morphism $char(km)$, so we have to show that the following diagram is a pullback square:

$$\begin{tikzcd}
S \times 1 \arrow[rr] \arrow[d, "m \times Id"] & & 1 \arrow[dd, "true"] \\
B' \times 1 \arrow[d, "k \times Id"] & & \\
B \times 1 \arrow[r, "Id \times \lceil char(m) \rceil"] & B \times \mathbb{P}B' \arrow[r, "e_k"] & \Omega
\end{tikzcd}$$

We have (immediatly) that the following square is a pullback square:

$$\begin{tikzcd}
B' \times 1 \arrow[r, "Id \times \lceil char(m) \rceil)"] \arrow[d, "k \times Id"] & B' \times \mathbb{P}B' \arrow[d, "k \times Id"] \\
B \times 1 \arrow[r, "Id \times \lceil char(m) \rceil)"] & B \times \mathbb{P}B'
\end{tikzcd}$$

This together with the pullback which defines e_k and U_B', we get a commutative diagram

$$\begin{tikzcd}
S \times 1 \arrow[d, "m \times Id"] & U \arrow[r] \arrow[d, "u_{B'}"] & 1 \arrow[r, "Id"] \arrow[d, "true"] & 1 \arrow[dd, "true"] \\
B' \times 1 \arrow[r, "Id \times \lceil char(m) \rceil"] \arrow[d, "k \times Id"] & B' \times \mathbb{P}B' \arrow[r, "\in_{B'}"] \arrow[d, "k \times Id"] & \Omega & \\
B \times 1 \arrow[r, "Id \times \lceil char(m) \rceil"] & B \times \mathbb{P}B' \arrow[rr, "e_k"] & & \Omega
\end{tikzcd}$$

where the left-below and both right diagrams are pullback squares. Since the right squares (either of one them is sufficient) is a pullback square, there exists a unique morphism $w : S \times 1 \to U_{B'}$ which completes the diagram. So to show that $S \times 1$ is the pullback of the whole diagram, it is sufficient to show that the following diagram is a pullback square (here we use that the other squares are pullback squares):

$$\begin{tikzcd}
S \times 1 \arrow[r, "w"] \arrow[d, "m \times Id"] & U_{B'} \arrow[r] \arrow[d, "u_{B'}"] & 1 \arrow[d, "true"] \\
B' \times 1 \arrow[r, "Id \times \lceil char(m) \rceil"] & B' \times \mathbb{P}B' \arrow[r, "\in_{B'}"] & \Omega
\end{tikzcd}$$

Let $u : X \to B'$ be a morphism such that $\in_{B'} \circ (Id \times \lceil char(m) \rceil) \circ (u \times Id) = true_X$, we have to show that u factors uniquely through m. But

$$char(m) \circ u = \in_{B'} \circ (Id \times \lceil char(m) \rceil) \circ (u \times Id) = true_X.$$

So by definition of the characteristic morphism of m (which is defined by a pullback), u factors indeed through m. $\square$

Theorem 2. *("**Beck-Chevally Condition for** $\exists$") Let $k : B' \to B$ be a monomorphism. If the diagram*

$$\begin{array}{ccc} C' & \xrightarrow{g'} & B' \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle k} \\ C & \xrightarrow{g} & B \end{array}$$

is a pullback square, then is the following diagram commutative:

$$\begin{array}{ccc} \mathbb{P}B' & \xrightarrow{\mathbb{P}g'} & \mathbb{P}C' \\ \downarrow{\scriptstyle \exists_k} & & \downarrow{\scriptstyle \exists_m} \\ \mathbb{P}B & \xrightarrow{\mathbb{P}g} & \mathbb{P}C \end{array}$$

Proof. We have to show $\mathbb{P}g \circ \exists_k = \exists_m \circ \mathbb{P}g' : \mathbb{P}B' \to \mathbb{P}C$. This is equivalent to showing that the $\mathbb{P}$-transpose of both morphisms are equal, i.e. we have to show

$$e_k(g \times Id) = e_m(Id \times \mathbb{P}g') : C \times \mathbb{P}B' \to \Omega.$$

So it is sufficient to show that these characteristic functions characterize the same subobjects of $C \times \mathbb{P}B'$. The corresponding subobjects are defined by the pullbacks of

$$C \times \mathbb{P}B' \xrightarrow{e_k(g\times Id)} \Omega \xleftarrow{true} 1$$
$$C \times \mathbb{P}B' \xrightarrow{e_m(Id\times \mathbb{P}g')} \Omega \xleftarrow{true} 1$$

Since a subobject has a unique characteristic morphism, we have to show that the (objects of the) pullbacks are isomorphic.
By the given we have that the left diagram is a pullback square and we have (immediatly) that the right diagram is also a pullback square:

$$\begin{array}{ccc} C' \times \mathbb{P}B' & \xrightarrow{g'\times Id} & B' \times \mathbb{P}B' \\ \downarrow{\scriptstyle m\times Id} & & \downarrow{\scriptstyle k\times Id} \\ C \times \mathbb{P}B' & \xrightarrow{g\times Id} & B \times \mathbb{P}B' \end{array} \qquad \begin{array}{ccc} C' \times \mathbb{P}B' & \xrightarrow{Id\times \mathbb{P}g'} & C' \times \mathbb{P}C' \\ \downarrow{\scriptstyle m\times Id} & & \downarrow{\scriptstyle m\times Id} \\ C \times \mathbb{P}B' & \xrightarrow{Id\times \mathbb{P}g'} & C \times \mathbb{P}C' \end{array}$$

By definition of $\in_k$ (resp. $\in_m$), we have that the following diagrams are pullback squares:

$$\begin{array}{ccccc} U_{B'} & \longrightarrow & 1 & \xrightarrow{Id} & 1 \\ \downarrow{\scriptstyle u_{B'}} & & \downarrow{\scriptstyle true} & & \\ B' \times \mathbb{P}B' & \xrightarrow{\in_{B'}} & \Omega & & \downarrow{\scriptstyle true} \\ \downarrow{\scriptstyle k\times Id} & & & & \\ B \times \mathbb{P}B' & & \xrightarrow{e_k} & & \Omega \end{array} \qquad \begin{array}{ccccc} U_{C'} & \longrightarrow & 1 & \xrightarrow{Id} & 1 \\ \downarrow{\scriptstyle u_{C'}} & & \downarrow{\scriptstyle true} & & \\ C' \times \mathbb{P}C' & \xrightarrow{\in_{C'}} & \Omega & & \downarrow{\scriptstyle true} \\ \downarrow{\scriptstyle m\times Id} & & & & \\ C \times \mathbb{P}C' & & \xrightarrow{e_m} & & \Omega \end{array}$$

Since the previous diagrams are pullbacks, we have that the pullback of

$$C \times \mathbb{P}B' \xrightarrow{e_k(g\times Id)} \Omega \xleftarrow{true} 1$$

is the pullback of

$$\begin{array}{ccccccc} & & U_{B'} & \longrightarrow & 1 & \xrightarrow{Id} & 1 \\ & & \downarrow{\scriptstyle u_{B'}} & & \downarrow{\scriptstyle true} & & \\ C' \times \mathbb{P}B' & \xrightarrow{g'\times Id} & B' \times \mathbb{P}B' & \xrightarrow{\in_{B'}} & \Omega & & \downarrow{\scriptstyle true} \\ \downarrow{\scriptstyle m\times Id} & & \downarrow{\scriptstyle k\times Id} & & & & \\ C \times \mathbb{P}B' & \xrightarrow{g\times Id} & B \times \mathbb{P}B' & & \xrightarrow{e_k} & & \Omega \end{array}$$

So the pullback is the pullback of the following diagram:

$$\begin{tikzcd} & U_{B'} \arrow[r] \arrow[d, "u_{B'}"] & 1 \arrow[d, "true"] \\ C' \times \mathbb{P}B' \arrow[r, "g' \times Id"] & B' \times \mathbb{P}B' \arrow[r, "\in_{B'}"] & \Omega \end{tikzcd}$$

Analogously is the pullback of

$$C \times \mathbb{P}B' \xrightarrow{e_m(Id \times \mathbb{P}g')} \Omega \xleftarrow{true} 1$$

given by the pullback of

$$\begin{tikzcd} & U_{C'} \arrow[r] \arrow[d, "u_{C'}"] & 1 \arrow[d, "true"] \\ C' \times \mathbb{P}B' \arrow[r, "Id \times \mathbb{P}g'"] & C' \times \mathbb{P}C' \arrow[r, "\in_{C'}"] & \Omega \end{tikzcd}$$

By the definition of $\mathbb{P}g'$, we have that $\in_{B'} \circ (g' \times 1) = \in_{C'} \circ (Id \times \mathbb{P}g')$, so the diagrams have the same pullback. □

Corollary 7. *If $k : B \to B'$ is a monomorphism, then is*

$$\mathbb{P}B \xrightarrow{\exists_k} \mathbb{P}B' \xrightarrow{\mathbb{P}k} \mathbb{P}B,$$

the identity on $\mathbb{P}B$.

Proof. We have that

$$\begin{tikzcd} B \arrow[r, "Id"] \arrow[d, "Id"] & B \arrow[d, "k"] \\ B \arrow[r, "k"] & B' \end{tikzcd}$$

is a pullback square. So by Beck-Chevally, we have that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}B \arrow[r, "\mathbb{P}Id"] \arrow[d, "\exists_k"] & \mathbb{P}B \arrow[d, "\exists_{Id}"] \\ \mathbb{P}B' \arrow[r, "\mathbb{P}k"] & \mathbb{P}B \end{tikzcd}$$

So $\mathbb{P}k \circ \exists_k = \exists_{Id} \circ \mathbb{P}Id$. Since $\mathbb{P}$ is a functor, we have $\mathbb{P}Id_B = Id_{\mathbb{P}B}$. So it remains to compute $\exists_{Id}$. Recall that $\exists_{Id_B}$ is the $\mathbb{P}$-transpose of e_{Id_B} and e_{Id_B} is defined as the characteristic morphism of $(Id_B \times Id) \circ u_B$, which was constructed as followed:

$$\begin{tikzcd} U_B \arrow[r] \arrow[d, "u_B"] & 1 \arrow[r, "Id"] \arrow[d, "true"] & 1 \arrow[dd, "true"] \\ B \times \mathbb{P}B \arrow[r, "\in_B"] \arrow[d, "Id_B \times Id"] & \Omega & \\ B \times \mathbb{P}B \arrow[rr, "e_{Id_B}"] & & \Omega \end{tikzcd}$$

where the upperleft square is also a pullback, so we see that $e_{Id} = \in_B$. So $\exists_{Id_B}$ is the $\mathbb{P}$-transpose of $\in_B$. Thus $\exists_{Id_B}$ is the unique morphism such that the following diagram commutes:

$$\begin{tikzcd} B \times \mathbb{P}B \arrow[rr, "Id_B \times \exists_{Id_B}"] \arrow[dr, "\in_B"] & & B \times \mathbb{P}B \arrow[dl, "\in_B"] \\ & \Omega & \end{tikzcd}$$

But Id_B satisfies also this condition (when replaced with $\exists_{Id_B}$) Thus we have $\exists_{Id} = Id$, so $\mathbb{P}k \circ \exists_k = \exists_{Id} \circ \mathbb{P}Id$ = Id. □

5 Monads and monadic functors

In this section we introduce the notion of monads and algebras on monads which are generalisations of the notion of a monoid and the action of a monoid on a set. Then we introduce monadic functors, these are functor which are *equivalent* to the forgetfull functor from algebras to the monad.

5.1 Monads

Recall that a monoid is a set M with a binary operation $\star : M \times M \to M$ which is associative and which has a neutral element for $\star$. Given a set M, one can always construct the free monoid on M as follows: Let

$$T(M) = \bigcup_{n \in \mathbb{N}} M^n = \bigcup_{n \in \mathbb{N}} \{(m_1, \cdots, m_n) | m_i \in M\},$$

i.e. $T(M)$ consists of all finite sequences with elements in M. Notice that every element $m \in M$ corresponds with the sequence (m) of 1 element, so one has a function

$$\epsilon_M : M \to T(M) : m \mapsto (m).$$

If $m := (m_1, \cdots, m_n)$ and $\tilde{m} := (\tilde{m}_1, \cdots, \tilde{m}_{\tilde{n}})$ are elements in $T(M)$, one can define the composition $m \star \tilde{m}$ as follows:

$$m \star \tilde{m} := (m_1, \cdots, m_n, \tilde{m}_1, \cdots, \tilde{m}_{\tilde{n}}).$$

Since an element in $T(M) \times T(M)$ can be considered as an element in $T(T(M))$, we have that the multiplication can be seen as a function:

$$\mu_M : T(T(M)) \to T(M) : (m, \tilde{m}) \mapsto m \star \tilde{m}.$$

So if one has already a monoid structure on M, one has a function

$$\eta_M : T(M) \to M : (m_1, \cdots, m_n) \mapsto m_1 m_2 \cdots m_n$$

and the empty sequence is mapped to the neutral element. If $f : M_1 \to M_2$ is a function (between sets), this induces a function

$$T(f) : T(M_1) \to T(M_2) : (m_1, \cdots, m_n) \mapsto (f(m_1), \cdots, f(m_n)).$$

So T defines a functor in $\mathbf{Set} \to \mathbf{Set}$ and μ_M and η_M defines natural transformations which satisfy certain properties corresponding with the neutral element and associativity. So a monoid can be generalized to the notion of a *monad*:

Definition 9. *A* ***monad*** *in a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\mu : T^2 \to T$ and $\eta : I \to T$ where I is the identityfunctor such that the following diagrams commute:*

$$\begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} IT \arrow[r, "\eta T"] \arrow[dr, "Id"'] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[dl, "Id"] \\ & T & \end{tikzcd}$$

The following proposition shows that each adjoint pair induces a monad:

Proposition 15. *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be an adjoint pair, with unit $\eta : Id_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to Id_{\mathcal{D}}$. Define*

$$\mu_C := G\epsilon_{F(C)} : GFGF(C) \to GF(C)$$

for each $C \in \mathcal{C}$. Then (GF, η, μ) defines a monad in $\mathcal{C}$.

Proof. We first have to show that μ_C is a natural transformation. Because $F(C) \in \mathcal{D}$ and ϵ is a natural transformation, we have for each $f \in Hom_{\mathcal{C}}(C_1, C_2)$ that the following diagram commutes:

$$\begin{array}{ccc} FGF(C_1) & \xrightarrow{\epsilon_{F(C_1)}} & F(C_1) \\ \big\downarrow{\scriptstyle FGF(f)} & & \big\downarrow{\scriptstyle F(f)} \\ FGF(C_2) & \xrightarrow{\epsilon_{F(C_2)}} & F(C_2) \end{array}$$

Since G is a functor, it preserves composition, so the following diagram commutes:

$$\begin{array}{ccc} GFGF(C_1) & \xrightarrow{G\circ\epsilon_{F(C_1)}} & GF(C_1) \\ \big\downarrow{\scriptstyle GFGF(f)} & & \big\downarrow{\scriptstyle GF(f)} \\ GFGF(C_2) & \xrightarrow{\epsilon_{GF(C_2)}} & GF(C_2) \end{array}$$

This means exactly that $T := GF$ is a natural transformation.
So we now show that (GF, η, μ) indeed defines a monad in $\mathcal{C}$.
We first show the associativity rule. Since ϵ is a natural transformation, we have that the following diagram commutes (for $C \in \mathcal{C}$, thus $F(C) \in \mathcal{D}$):

$$\begin{array}{ccc} FG(FG(FC)) & \xrightarrow{\epsilon_{FG(FC)}} & Id(FG(FC)) \\ \big\downarrow{\scriptstyle FG(\epsilon_{FC})} & & \big\downarrow{\scriptstyle \epsilon_{FC}} \\ FG(FC) & \xrightarrow{\epsilon_{FC}} & Id(FC) \end{array}$$

So by precomposing with G we have that the following diagram commutes (because G is a functor and thus preserves composition):

$$\begin{array}{ccc} T^3(C) = GFG(FG(FC)) & \xrightarrow{G\epsilon_{FG(FC)}} & GFG(FC) = T^2(C) \\ \big\downarrow{\scriptstyle GFG(\epsilon_{FC})} & & \big\downarrow{\scriptstyle G\epsilon_{FC}} \\ T^2(C) = GFG(FC) & \xrightarrow{G\epsilon_{FC}} & GFC = T(C) \end{array}$$

Since $\mu_C := G\epsilon_{FC}$, we thus have the associativity.
Because η, ϵ form an adjunction, we have

$$Id_{F(C)} = \epsilon_{F(C)} \circ F(\eta_C), \quad Id_{G(D)} = G(\epsilon_D) \circ \eta_{G(D)}.$$

So in particular by composing with G resp. F:

$$Id_{GF(C)} = G\epsilon_{F(C)} \circ GF\eta_C, \quad Id_{FG(D)} = FG(\epsilon_{FD}) \circ F\eta_{G(D)},$$

which are exactly the *axioms corresponding to the neutral element.* □

5.2 Algebras on a monad

A monoid M (with multiplication $\nu : M \times M \to M$ and neutral element given by $e : 1 \to M$) acts on a set Y if there is a function $h : M \times Y \to Y$ such that the following diagrams commute:

$$\begin{array}{ccc} M \times M \times Y & \xrightarrow{Id_M \times h} & M \times Y \\ \big\downarrow{\scriptstyle \nu \times Id_Y} & & \big\downarrow{\scriptstyle h} \\ M \times Y & \xrightarrow{h} & Y \end{array} \qquad \begin{tikzcd} Y \cong 1 \times Y \arrow[r, "e \times Id_Y"] \arrow[dr, "Id_Y"] & M \times Y \arrow[d, "h"] \\ & Y \end{tikzcd}$$

Definition 10. *Let (T, μ, η) be a monad on a category $\mathcal{C}$. The category of T-algebras, denoted by $\mathcal{C}^T$, has as its objects pairs $(C, h : TC \to C)$ where $C \in \mathcal{C}$ an object and h a morphism such that the following diagrams commute:*

$$\begin{array}{ccc} T^2C & \xrightarrow{Th} & TC \\ \big\downarrow{\scriptstyle \mu_C} & & \big\downarrow{\scriptstyle h} \\ TC & \xrightarrow{h} & C \end{array} \qquad \begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "Id"] & TC \arrow[d, "h"] \\ & C \end{tikzcd}$$

A morphism $f : (C_1, h_1) \to (C_2, h_2)$ of T-algebras is a morphism $f : C_1 \to C_2$ such that the following diagram commutes:

$$\begin{array}{ccc} TC_1 & \xrightarrow{Tf} & TC_2 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ C_1 & \xrightarrow{f} & C_2 \end{array}$$

The following lemma is clear:

Lemma 3. *Let (C, h) be a (T, η, μ)-algebra. The assignment $(C, h) \mapsto C$ induces a functor*

$$G^T : \mathcal{C}^T \to \mathcal{C},$$

called the forgetfull functor.

Proposition 16. *Let (T, η, μ) be a monad in $\mathcal{C}$. The forgetfull functor*

$$G^T : \mathcal{C}^T \to \mathcal{C} : (C, h) \mapsto C,$$

has a left adjoint.

Proof. Define $F^T : \mathcal{C} \to \mathcal{C}^T$ by $F^T C := (TC, \mu_C)$. That $F^T C$ is indeed a T-algebra is by definition of μ_C since (T, η, μ) is a monad. For $f \in Hom_{\mathcal{C}}(C_1, C_2)$, set $F^T(f) := Tf$. Because $\mu : T^2 \to T$ is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc} T^2C_1 & \xrightarrow{\mu_{C_1}} & TC_1 \\ \downarrow{\scriptstyle T^2 f} & & \downarrow{\scriptstyle TF} \\ T^2C_2 & \xrightarrow{\mu_{C_2}} & TC_2 \end{array}$$

Which means that

$$Tf : (TC_1, \mu_{C_1}) \to (TC_2, \mu_{C_2}),$$

is a T-algebra morphism. Since T is a functor, so is F^T.
We now show that $F^T \dashv G^T$. Notice that for $C \in \mathcal{C}$,

$$G^T F^T C = G^T(TC, \mu_C) = TC,$$

thus a natural transformation $\eta^T : Id_{\mathcal{C}} \to G^T F^T$ consists of morphisms $C \to TC$, so define $\eta^T := \eta$ (this is a natural transformation because η is).
For $(C, h) \in \mathcal{C}^T$, we have

$$F^T G^T(C, h) = F^T C = (TC, \mu_C),$$

thus a natural transformation $\epsilon^T : F^T G^T \to Id_{\mathcal{C}^T}$ consists of morphisms $(TC, \mu_C) \to (C, h)$. Since (T, h) is a T-algebra, the following diagram commutes:

$$\begin{array}{ccc} T^2C & \xrightarrow{Th} & TC \\ \downarrow{\scriptstyle \mu_C} & & \downarrow{\scriptstyle h} \\ TC & \xrightarrow{h} & C \end{array}$$

This shows that

$$h : (TC, \mu_T) \to (C, h),$$

is a morphism of T-algebras and in particular that

$$\epsilon^T := \Big((TC, \mu_C) \xrightarrow{h} (C, h)\Big)_{(C,h)},$$

is a natural transformation. So we are now left to show

$$Id_{F^TC} = \epsilon^T_{F^TC} \circ F^T(\eta^T_C), \quad Id_{G^T(C,h)} = G^T(\epsilon_{(C,h)}) \circ \eta_{G^T(C,h)}.$$

From

$$\begin{array}{rcl} \epsilon^T_{F^TC} = \epsilon_{(TC,\mu_C)} = \mu_C & , & F^T(\eta^T_C) = F^T(\eta_C) = T\eta_C \\ G^T(\epsilon^T_{(C,h)} = G^T(h) = h & , & \eta^T_{G^T(C,h)} = \eta^T_C = \eta_C, \end{array}$$

the equations we have to show reduce to

$$\mu_C \circ T\eta_C = Id, \quad h \circ \eta_C = Id$$

By definition of (T, μ, η) being a monad, the first equation holds and by definition of (C, h) being a T-algebra, the second equation holds. So η (resp. ϵ) is indeed a unit (resp. counit) of an adjunction $F^T \dashv G^T$. □

We have seen that we have a *forgetfull* functor from the algebras over a monad to the monad itself. We call a functor (with a left adjoint) monadic, if it is in a sense equivalent to the forgetfull functor induced by adjointness.

Proposition 17. *Let $F \dashv: \mathcal{C} \to \mathcal{D}$ be an adjoint pair. Then there exists a functor*

$$K : \mathcal{D} \to \mathcal{C}^T$$

such that the following diagram (of categories and functors) commute:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, shift left, "G"] & \mathcal{C}^T \arrow[d, shift right, "G^T"'] \\ \mathcal{C} \arrow[u, shift left, "F"] \arrow[r, "Id_{\mathcal{C}}"] & \mathcal{C} \arrow[u, shift right, "F^T"'] \end{tikzcd}$$

We call K the ***comparison functor****.*

Proof. Let $(T,)$ be the monad induced by the adjunction. Define for $D \in \mathcal{D}$:

$$K(D) := (GD, G\epsilon_D).$$

Since $\epsilon : FG \to Id_{\mathcal{D}}$ is a natural transformation and $\epsilon_D \in Hom_{\mathcal{D}}(FGD, D)$, we have that the following diagram commutes:

$$\begin{tikzcd} FG(FGD) \arrow[r, "\epsilon_{FGD}"] \arrow[d, "FG(\epsilon_D)"] & Id_{\mathcal{D}}(FGD) = FG(D) \arrow[d, "\epsilon_D"] \\ FGD \arrow[r, "\epsilon_D"] & Id_{\mathcal{D}}(D) \end{tikzcd}$$

Since G is a functor, it preserves composition. So applying this diagram to G gives the desired commutative diagram which shows the first requirement to be a T-algebra. The second requirement is that the following diagram commutes:

$$\begin{tikzcd} GD \arrow[r, "\eta_{GD}"] \arrow[dr, "Id_{GD}"] & GF(GD) \arrow[d, "G\epsilon_{GD}"] \\ & GD \end{tikzcd}$$

But this commutativity holds because (F, G) form an adjoint pair with unit η and counit ϵ. So $K(D)$ is indeed a T-algebra.
For a morphism $f \in Hom_{\mathcal{D}}(D_1, D_2)$, set $K(f) := G(f)$. We have to show that

$$K(f) = G(f) : (GD_1, G\epsilon_{D_1}) \to (GD_2, G\epsilon_{D_2}),$$

is a morphism of T-algebras. Because ϵ is a natural transformation and $G(f) \in Hom_{\mathcal{C}}(GD_1, GD_2)$, we have that the following diagram commutes:

$$\begin{tikzcd} FG(D_1) \arrow[r, "\epsilon_{D_1}"] \arrow[d, "FG(f)"] & GD_1 \arrow[d, "G(f)"] \\ FG(D_2) \arrow[r, "\epsilon_{D_2}"] & GD_2 \end{tikzcd}$$

So by applying this commutative diagram to the functor G, we get the desired commutative diagram to show that $G(f)$ is indeed a morphism of T-algebras. Since G is a functor, we have that K is a functor. So we are left to show that

$$\begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, shift right, "G"] & \mathcal{C}^T \arrow[d, shift left, "G^T"'] \\ \mathcal{C} \arrow[u, shift right, "F"] \arrow[r, "Id_{\mathcal{C}}"] & \mathcal{C} \arrow[u, shift left, "F^T"'] \end{tikzcd}$$

commutes. That $G^T \circ K = G$ follows from:

$$\begin{aligned} G^T \circ K(D) &= G^T(GD, G\epsilon_D) = G(D) \\ G^T \circ K(f) &= G^T(K(f)) = G(f) \end{aligned}$$

That $K \circ F = F^T$ follows from:

$$\begin{aligned} K \circ F(C) &= K(F(C)) = (GFC, G\epsilon_{FC}) = (TC, \mu_C) = F^T(C) \\ K \circ F(f) &= K(F(f)) = GF(f) = T(f) = F^T(f) \end{aligned}$$

□

Definition 11. *A functor $G : \mathcal{A} \to \mathcal{C}$ is **monadic** if G has a left adjoint F and the comparison functor $K : \mathcal{A} \to \mathcal{C}^T$ (defined in the previous proposition) is an equivalence of categories.*

Definition 12. *A functor $F : \mathcal{A} \to \mathcal{B}$ **creates limits** if for each diagram $H : J \to \mathcal{A}$ (i.e. a functor) such that $F \circ H$ has a limiting cone $(B \xrightarrow{q_j} FH(j))_{j \in J}$, there exists a cone $(A \xrightarrow{p_j} H(j))_{j \in J}$ (on H) such that $F(A) = B$ and $F(p_j) = q_j$ for all $j \in J$ and this cone is a limiting cone on H.*

Theorem 3. *A monadic functor $G : \mathcal{A} \to \mathcal{C}$ creates limits.*

Proof. If $G : \mathcal{A} \to \mathcal{C}$ is monadic, there is an equivalence of categories $K : \mathcal{A} \to \mathcal{C}^T$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "K"] \arrow[dr, "G"'] & \mathcal{C}^T \arrow[d, "G^T"] \\ & \mathcal{C} \end{tikzcd}$$

As an equivalence of categories preserves limits, it therefore suffices to show that G^T preserves limits.
Let $H : J \to \mathcal{C}^T$ be a diagram and write $H(j) =: (C_j, TC_j \xrightarrow{h_j} C_j)$. Assume that

$$\tau := (C \xrightarrow{\tau_j} G^T \circ H(j))_{j \in J},$$

is a limiting cone (on $G^T \circ H$). Since G^T is the forgetfull functor, we have $G^T \circ H(j) = C_j$.
Let $\phi \in Hom_J(j_1, j_2)$. Consider the following diagram:

$$\begin{tikzcd} & TC \arrow[dl, "T\tau_{j_1}"'] \arrow[dr, "T\tau_{j_2}"] & \\ TC_{j_1} \arrow[rr, "TG^T H(\phi)"] \arrow[d, "h_{j_1}"] & & TC_{j_2} \arrow[d, "h_{j_2}"] \\ G^T H(j_1) = C_{j_1} \arrow[rr, "G^T H(\phi)"] & & C_{j_2} = G^T H(j_2) \end{tikzcd}$$

The (top) triangle commutes because τ is a natural transformation (so $G^T H(\phi) \circ \tau_{j_1} = \tau_{j_2}$) and then applying T preserves the composition. The (bottom) square commutes because $G^T H(\phi)$ is a T-algebra morphism $(C_{j_1}, h_{j_1}) \to (C_{j_2}, h_{j_2})$. This shows that

$$(TC \xrightarrow{h_j \circ T\tau_j} G^T \circ H(j) = C_j)_{j \in J},$$

is a cone on $G^T \circ H$. But τ is the limiting cone, so there exists a unique morphism $h : TC \to C$ such that for each $j \in J$, the following diagram commutes:

$$\begin{array}{ccc} TC & \xrightarrow{h} & C \\ \Big\downarrow{\scriptstyle T\tau_j} & & \Big\downarrow{\scriptstyle \tau_j} \\ TC_j & \xrightarrow{h_j} & C_j \end{array}$$

We now show that (C, h) is a T-algebra, i.e. we have to show that

$$h \circ \eta_C = Id_C, \quad h \circ \mu_C = h \circ TH.$$

Consider the following diagram:

$$\begin{array}{ccccc} C & \xrightarrow{\eta_C} & TC & \xrightarrow{h} & C \\ \Big\downarrow{\scriptstyle \tau_j} & & \Big\downarrow{\scriptstyle T\tau_j} & & \Big\downarrow{\scriptstyle \tau_j} \\ C_j & \xrightarrow{\eta_{C_j}} & TC_j & \xrightarrow{h_j} & C_j \end{array}$$

The left square commutes by naturality of η and the right square commutes by h. Since (C_j, h_j) is a T-algebra, the bottom row is the identity Id_{C_j}. So for all $j \in J$,

$$\tau_j = \tau_j \circ h \circ \eta_C.$$

So by the universal property of the limiting cone τ, we have $Id_C = h \circ \eta_C$.
Using the commutativity of the right commuting square, we have

$$\begin{aligned} \tau_j \circ h \circ Th &= h_j \circ T(\tau_j) \circ T(h) = h_j \circ T(\tau_j \circ h) \\ &= h_j \circ T(h_j \circ T(\tau_j)) = h_j \circ T(h_j) \circ T^2(\tau_j) \end{aligned}$$

Now consider the following diagram:

$$\begin{array}{ccccc} T^2C & \xrightarrow{T^2(\tau_j)} & T^2C_j & \xrightarrow{T(h_j)} & TC_j \\ \Big\downarrow{\scriptstyle \mu_C} & & \Big\downarrow{\scriptstyle \mu_{C_j}} & & \Big\downarrow{\scriptstyle h_j} \\ TC & \xrightarrow{T(\tau_j)} & TC_i & \xrightarrow{h_j} & C_j \end{array}$$

The left square commutes by the naturality of μ and the right commutes because (C_j, h_j) is a T-algebra. So we have

$$\begin{aligned} \tau_j \circ h \circ Th &= h_j \circ T(h_j) \circ T^2(\tau_j) \\ &= h_j \circ T(\tau_j) \circ \mu_C \\ &= \tau_j \circ h \circ \mu_C \end{aligned}$$

Again by the universal property of the limiting cone τ, we have $h \circ Th = h \circ \mu_C$. Thus (T, h) is indeed a T-algebra. So in particular we have that $\tau_j : C \to C_j$ is a morphism of T-algebras. Because $G^T : \mathcal{C}^T \to \mathcal{C}$ is faithfull, i.e. $Hom_{\mathcal{C}}^T(C_1, C_2) \subseteq Hom_{\mathcal{C}}(C_1, C_2)$, we have that

$$\sigma := (\tau_j : (C, h) \to (C_j, h_j) = H(j))_{j \in J}$$

is a cone on H and in particular $G^T \sigma = \tau$. And so we are now left to show that σ is a limiting cone of H.
Assume that

$$\kappa := \left(\kappa_j : (\tilde{C}, \tilde{h}) \to H(j) = (C_j, h_j)\right)_{j \in J},$$

is another cone of H. So κ is in particular a cone on $G^T H$. But τ is the limiting cone on $G^T H$, so there exists a unique morphism $f \in Hom_{\mathcal{C}}(\tilde{C}, C)$ such that $\kappa_j = \tau_j \circ f$. So we are left to show that

$$(\tilde{C}, \tilde{h}) \xrightarrow{f} (C, h),$$

is a morphism of T-algebras, i.e. $f \circ \tilde{h} = h \circ Tf$. Using the universal property of the limiting cone, this equality follows from:

$$\begin{aligned}
\tau_j \circ f \circ \tilde{h} &= \kappa \circ \tilde{h} = h_j \circ T(\kappa_j) \\
&= h_j \circ T(\tau_j \circ f) \\
&= h_j \circ T(\tau_j) \circ T(f) \\
&= \tau_j \circ h \circ T(f)
\end{aligned}$$

□

We now give a criteria for when a functor is monadic.

Theorem 4. *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be an adjoint pair, T its corresponding monad and $K : \mathcal{D} \to \mathcal{C}^T$ be the comparison functor. Then*

1. *If $\mathcal{D}$ has coequalizers of all reflexive pairs, then K has a left adjoint L.*
2. *If G preserves the coequalizers of the reflexive pairs, then is the unit of the adjunction $L \vdash K$ an isomorphism, i.e. $Id_{\mathcal{C}^T} \cong K \circ L$.*
3. *If G reflects isomorphisms, then the counit of $L \dashv K$ is an isomorphism, i.e. $L \circ K \cong Id_{\mathcal{D}}$.*

Proof. Let $(C, h : GFC \to C) \in \mathcal{C}^T$ be a T-algebra, we first construct $L(C, h)$: If ϵ (resp. η) is the counit (resp. unit) of $F \dashv G$, then $Id_{F(C)} = \epsilon_{F(C)} \circ F(\eta_C)$. And by definition of a T-algebra, we have $h \circ \eta_C = Id_C$, so $Fh \circ F\eta_C = Id_{F(C)}$. Thus $Fh, \epsilon_{FC} : FGFC \to FC$ is a reflexive pair, so by hypothesis it has a coequalizer $e : FC \to L(C, h)$, i.e. the following diagram is a coequalizer diagram:

$$\begin{tikzcd} FGFC \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\epsilon_{FC}"'] & FC \arrow[r, "e"] & L(C,h) \end{tikzcd}$$

Let $f \in Hom_{\mathcal{C}^T}((C_1, h_1), (C_2, h_2))$ is a morphism of T-algebras. So by the universal property of the coequalizer, there exists a unique morphism $L(f)$ such that the following diagram commutes:

$$\begin{tikzcd} FC_1 \arrow[r, "e_1"] \arrow[d, "F(f)"] & L(C_1, h_1) \arrow[d, "L(f)"] \\ F(C_2) \arrow[r, "e_2"] & L(C_2, h_2) \end{tikzcd}$$

It is immediate that L is a functor, because F is.
We now show that $L \dashv K$. We first define the unit λ. By definition of a T-algebra, we have that $h \circ \eta_C = Id_C$. So we have that the following diagram is a coequalizer diagram:

$$\begin{tikzcd} GFGFC \arrow[r, shift left, "GFh"] \arrow[r, shift right, "G\epsilon_{FC}"'] & GFC \arrow[r, "h"] & C \end{tikzcd}$$

As G preserves composition, we have that the following diagram commutes:

$$\begin{tikzcd} GFGFC \arrow[r, shift left, "GFh"] \arrow[r, shift right, "G\epsilon_{FC}"'] & GFC \arrow[r, "Ge"] & GL(C,h) \end{tikzcd}$$

So by the universal property of the equalizer, there exists a unique morphism

$$\lambda_C : C \to GL(C, h),$$

such that $Ge = \lambda_C \circ h$. We now claim that λ_C defines a natural transformation

$$\lambda : Id_{\mathcal{C}^T} \to K \circ L,$$

i.e. we have to show that for each morphism $f : (C_1, h_1) \to (C_2, h_2)$ of T-algebras, the following diagram commutes:

$$\begin{array}{ccc} (C_1, h_1) & \xrightarrow{\lambda_{C_1}} & KL(C_1, h_1) \\ \Big\downarrow f & & \Big\downarrow KL(f) \\ (C_2, h_2) & \xrightarrow{\lambda_{C_2}} & KL(C_2, h_2) \end{array}$$

Using that h_1 is an epimorphism (since it is a coequalizer), the equality follows from the following calculation:

$$\begin{aligned} \lambda_{C_2} \circ f \circ h_1 &= \lambda_{C_2} \circ h_2 \circ GF(f) \\ &= G(e_2) \circ GF(f) = G(e_2 \circ F(f)) \\ &= G(L(f) \circ e_1) = GL(f) \circ G(e_1) \\ &= GL(f) \circ \lambda_{C_1} \circ h_1 \\ &= KL(f) \circ \lambda_{C_1} \circ h_1 \end{aligned}$$

We now define the counit κ. By definition of L, we have for any $D \in \mathcal{D}$ that the following diagram is a coequalizer diagram:

$$FGFGD \underset{\epsilon_{FGD}}{\overset{FG\epsilon_D}{\rightrightarrows}} FGD \xrightarrow{e_{GD}} LKD = L(GD, G\epsilon_D)$$

But $\epsilon_D \circ FG\epsilon_D = \epsilon_D \circ \epsilon_{FGD}$, so by the universal property of the equalizer, there exists a unique morphism $\kappa_D : LKD \to D$ such that $\epsilon_D = \kappa_D \circ e_{GD}$. That $\kappa := (\kappa_D)_{D \in \mathcal{D}}$ is a natural transformation, follows from:

$$\begin{aligned} f \circ \kappa_{D_1} \circ e_{GD_1} &= f \circ \epsilon_{D_1} \\ &= \epsilon_{D_2} \circ FG(f) \\ &= \kappa_{D_2} \circ e_{GD_2} \circ FG(f) \\ &= \kappa_{D_2} \circ LG(f) \circ e_{GD_1} \\ &= \kappa_{D_2} \circ LK(f) \circ e_{GD_1} \end{aligned}$$

because e_{GD_1} is an epimorphism. So $L \dashv K$ as desired.
If G preserves the coequalizers of the reflexive pairs, then we have that Ge is the coequalizer of GFh with $G\epsilon_{FC}$, but $L(C, h)$ was by definition the coequalizer, so λ_C is an isomorphism (for all C). So $Id_{\mathcal{C}^T} \cong K \circ L$. Using η_{GD}, we have that

$$GFGFGD \underset{G\epsilon_{FGD}}{\overset{GFG\epsilon_D}{\rightrightarrows}} GFGD \xrightarrow{G\epsilon_D} GD$$

is a coequalizer diagram. So $G(\kappa_D)$ is an isomorphism, so if moreover, G reflects isomorphisms, we have that κ_D is an isomorphism (for all D), so $L \circ K \cong Id_{\mathcal{D}}$. □

So by the definition of a monadic functor, the previous theorem summarizes as follows:

Corollary 8. *(**"Beck's theorem"**) If $\mathcal{A}$ has coequalizers of all reflexive pairs, the functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, reflects isomorphism and preserves those coequalizers (of reflexive pairs), then is G monadic.*

6 Colimits

In this section we will show that each topos $\mathcal{E}$ has all finite colimits. This will be done by showing that $\mathcal{E}^{op}$ has finite limits.

Proposition 18. *The power set functor $\mathbb{P} : \mathcal{E}^{op} \to \mathcal{E}$ has a left adjoint.*

Proof. We claim that

$$\mathbb{P}^{op} : \mathcal{E} = (\mathcal{E}^{op})^{op} \to \mathcal{E}^{op},$$

is the left-adjoint of $\mathbb{P}$, indeed:

$$Hom_{\mathcal{E}}(A, \mathbb{P}B) \cong Hom_{\mathcal{E}}(A \times B, \Omega) \cong Hom_{\mathcal{E}}(B \times A, \Omega) \cong Hom_{\mathcal{E}}(B, \mathbb{P}A) \cong Hom_{\mathcal{E}^{op}}(\mathbb{P}A, B),$$

where we used that $\Omega^B \cong \mathbb{P}B$. □

Proposition 19. *The powerobject functor is faithfull.*

Proof. Let $h, k \in Hom(B, A)$ such that $\mathbb{P}h = \mathbb{P}k$, so

$$\mathbb{P}h \circ \{\cdot\}_A = \mathbb{P}k \circ \{\cdot\}_A.$$

Taking the $\mathbb{P}$-tranpose of this equation gives us

$$\delta_A \circ (h \times 1) = \delta_A \circ (k \times 1) : B \times A \to A \times A \to \Omega.$$

As this is a morphism to Ω, they classify the same subobject. But the following diagram is a pullback square:

$$\begin{tikzcd}
B \arrow[r, "h"] \arrow[d, "(Id,h)"] & A \arrow[r] \arrow[d, "\Delta_A"] & 1 \arrow[d, "true"] \\
B \times A \arrow[r, "(h,Id)"] & A \times A \arrow[r, "\delta_A"] & \Omega
\end{tikzcd}$$

And the same when h is replaced with k. So there exists an isomorphism $\phi \in Hom(X, X)$ such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[rr, "\phi"] \arrow[dr, "(h,Id)"'] & & B \arrow[dl, "(k,Id)"] \\
& B \times A &
\end{tikzcd}$$

So

$$(k \circ \phi, \phi) = (k, Id) \circ \phi = (h, Id).$$

Thus by projection on the second component we have $\phi = Id$, thus by projection on the first component we have

$$h = k \circ \phi = k \circ Id = k.$$

So $\mathbb{P}$ is indeed faithfull. □

Corollary 9. *The powerobject functor reflects isomorphisms.*

Proof. Because $\mathbb{P}$ is faithfull, it reflects mono -and epimorphisms. But a topos is balanced, i.e. an epimorphism which is a monomorphism is an isomorphism, so $\mathbb{P}$ reflects isomorphisms. □

Proposition 20. *The powerobject functor $\mathbb{P} : \mathcal{E}^{op} \to \mathcal{E}$ is monadic and in particular creates all limits.*

Proof. By Beck's theorem we have to show that $\mathcal{E}^{op}$ has coequalizers of all reflexive pairs, $\mathbb{P}$ preserves these coequalizers, $\mathbb{P}$ has a left adjoint and $\mathbb{P}$ reflects isomorphisms.
We have seen that $\mathbb{P}^{op}$ is the left adjoint of $\mathbb{P}$. In the previous corollary we have shown that is reflects isomorphisms. Since $\mathcal{E}$ has equalizers (of all pairs), $\mathcal{E}^{op}$ has (by definition of the opposite category) all coequalizers, so in particular of the reflexive pairs. So we only have to show that $\mathbb{P}$ preserves those coequalizers. So consider such a coequalizer in $\mathcal{E}^{op}$, i.e. we have the following equalizer diagram (in $\mathcal{E}$):

$$C \xrightarrow{g} B \underset{h}{\overset{k}{\rightrightarrows}} A$$

and because f and g form a reflexive pair, there exists a morphism $d : A \to B$ such that $d \circ h = Id_B = d \circ k$. So we have to show that the following diagram is a coequalizer (in $\mathcal{E}$):

$$\mathbb{P}A \underset{\mathbb{P}k}{\overset{\mathbb{P}h}{\rightrightarrows}} \mathbb{P}B \xrightarrow{\mathbb{P}g} \mathbb{P}C$$

Because $hg = hk$ and $\mathbb{P}$ is a functor, we have $\mathbb{P}g \circ \mathbb{P}h = \mathbb{P}g \circ \mathbb{P}k$, so we only have to show the universal property. Assume there is an object $D \in \mathcal{E}$ and a morphism $f \in Hom(\mathbb{P}B, D)$ such that the following diagram is commutative:

$$\mathbb{P}A \underset{\mathbb{P}k}{\overset{\mathbb{P}h}{\rightrightarrows}} \mathbb{P}B \xrightarrow{f} D$$

We have to show that f factorises through $\mathbb{P}g$.
Consider the following diagram:

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "g"] & B \arrow[d, "h"] \\ B \arrow[r, "k"] & A \end{tikzcd}$$

This diagram commutes and we now claim that it is a pullback square: Consider the following commuting diagram:

$$\begin{tikzcd} X \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & C \arrow[r, "g"] \arrow[d, "g"] & B \arrow[d, "h"] \\ & B \arrow[r, "k"] & A \end{tikzcd}$$

So in particular (by composing with d) we have

$$f_1 = Id_B \circ f_1 = d \circ h \circ f_1 = d \circ k \circ f_2 = f_2.$$

Thus $h \circ f_1 = k \circ f_2 = k \circ f_1$, but g is the equalizer of h and k, so there exists a unique $s : X \to C$ such that $f_2 = f_1 = g \circ s$. So it is indeed a pullback square.
Since g is an equalizer, it is a monomorphism. And because $dk = Id_B$, k is also a monomorphism (if $k\phi_1 = k\phi_2$, for some morphisms ϕ_1, ϕ_2, then $\phi_1 = dk\phi_1 = dk\phi_2 = \phi_2$). So we are now in the setting of Beck-Chevalley, thus the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}B \arrow[r, "\mathbb{P}g"] \arrow[d, "\exists_h"] & \mathbb{P}C \arrow[d, "\exists_g"] \\ \mathbb{P}A \arrow[r, "\mathbb{P}k"] & \mathbb{P}B \end{tikzcd}$$

In the same way that k is a monomorphism, h is also a monomorphism. Thus from the corollary of Beck-Chevalley, we have $\mathbb{P}g \circ \exists_g = Id_{\mathbb{P}C}$ and $\mathbb{P}h \circ \exists_h = Id_{\mathbb{P}B}$.
So we get:

$$f \circ \exists_g \circ \mathbb{P}g = f \circ \mathbb{P}k \circ \exists_h = f \circ \mathbb{P}h \circ \exists_h = f \circ Id_{\mathbb{P}B} = f.$$

This shows that f factorizes through $\mathbb{P}g$, thus $\mathbb{P}g$ is indeed the coequalizer of $\mathbb{P}h$ and $\mathbb{P}k$. Thus $\mathbb{P}$ preserves coequalizers of reflexive pairs. Thus $\mathbb{P}$ is monadic by Beck's theorem. $\square$

We are now ready to prove that a topos has all finite colimits:

Theorem 5. *A topos $\mathcal{E}$ is finitely cocomplete.*

Proof. To show that $\mathcal{E}$ has all colimits, we can equivalently show that $\mathcal{E}^{op}$ has all finite limits. Let J be a finite category and consider a J-diagram $H : J \to \mathcal{E}^{op}$. Because $\mathcal{E}$ has all limits, we know that $\mathbb{P} \circ H$ has a limiting cone. But $\mathbb{P}$ creates limits, thus there exists a limiting cone for H which shows that $\mathcal{E}^{op}$ has all finite limits. $\square$